Westfälische Wilhelms-Universität Münster

Applied Mathematics Münster

ArbiLoMod: Communication Avoiding Localized Reduced Basis Methods for Problems with Arbitrary Local Modifications

MoRePaS 2015 - Trieste

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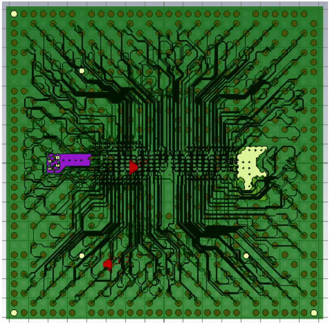
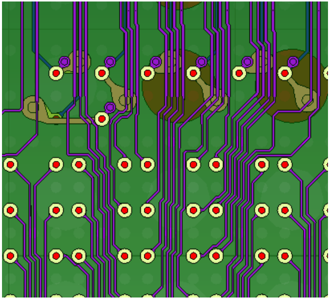
Christian Engwer, Mario Ohlberger, Stephan Rave

October 14, 2015

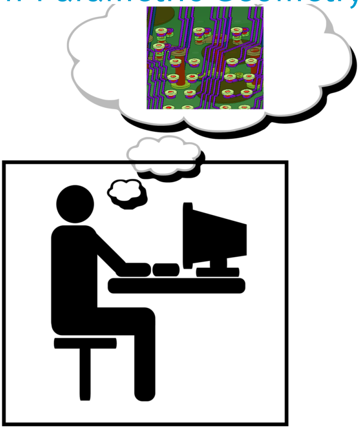
Motivation: Chip Carrier in a Flip Chip Package

Motivation: Chip Carrier, interior





Non-Parametric Geometry Changes



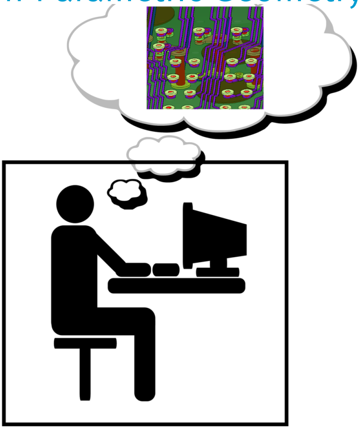
Multi-query setting:
Sequence of geometries

Properties of changes:

1. very localized
2. unforeseen

Cluster often available.
Cloud always available.

Non-Parametric Geometry Changes



Multi-query setting:
Sequence of geometries

Properties of changes:

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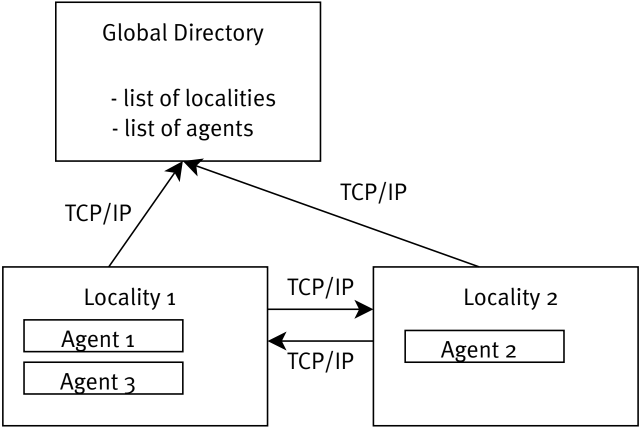
Cluster often available.
Cloud always available.

Goal: time harmonic Maxwell's equations
Examples here: heat conduction

Attacking from two Sides

1. Method Design
   - Localized Model Order Reduction
     ... like PR-SCRBE, GMsFEM, LRBMS, GFEM
2. Software Design
   - Event driven design
   - Task based parallelism
   - Agent based implementation
     ... like HPX, ViennaX

Event Driven, Agent Based Implementation



Setting

- Coercive, continuous, parametric bilinear form

$$a : V \times V \times \mathcal{P} \to \mathbb{R}$$

$$\alpha_{LB} ||u||_V^2 \leq a(u, u, \mu) \qquad \forall \mu \in \mathcal{P}$$

- Linear, parametric form

$$f : V \times \mathcal{P} \to \mathbb{R}$$

- Problem: find u_μ in V such that

$$a(u_\mu, v; \mu) = f(v; \mu) \qquad \forall v \in V$$

Main Idea

- Modify the Reduced Basis Method (RBM) to use basis functions with local support.
- After geometry change, reuse basis functions which do not have support in changed area.

ArbiLoMod Ingredients

1. Localized spaces
2. Localized basis generation (training and greedy)
3. Localized a-posteriori error estimation
4. Localized adaptive basis enrichment

Localized Reduced Basis

Space decomposition into local subspaces

$$V = \bigoplus_i V_i \qquad V_i \subset V$$

Find reduced local spaces

$$\widetilde{V}_i \subset V_i$$

Construct global reduced space

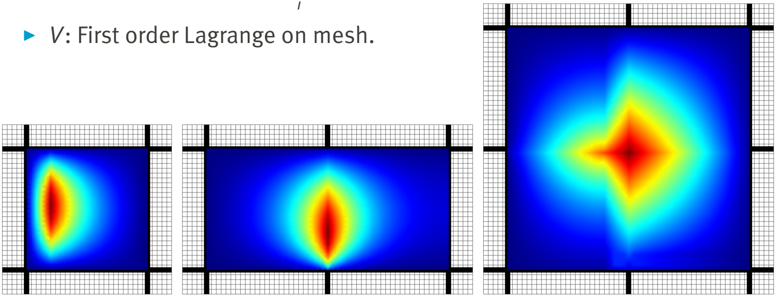
$$\widetilde{V}_{\mathrm{LRB}} := \bigoplus_i \widetilde{V}_i$$

Space Decomposition

- Nonoverlapping Domain Decomposition:

$$\Omega = \bigcup_i \Omega_i \qquad \Omega_i \cap \Omega_j = \emptyset \quad i \neq j$$

- V: First order Lagrange on mesh.



Codim-0 Space   Codim-1 Space   Codim-2 Space

mesh line
domain boundary

Projection Operators

- Direct decomposition of ansatz space:

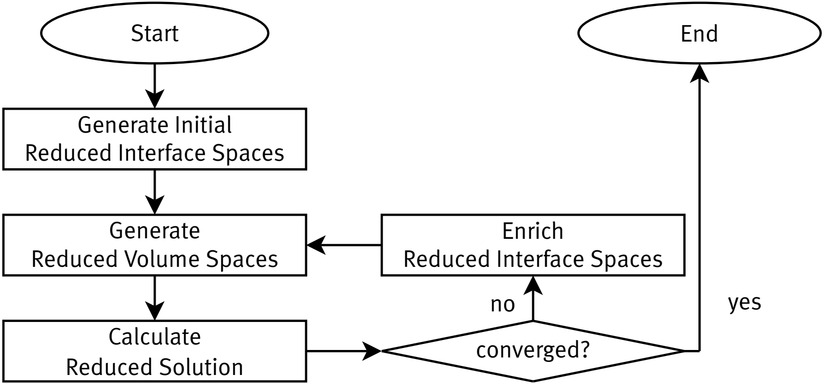
$$V = \bigoplus_i V_i \qquad V_i \subset V$$

- Definition of projection operators:

$$P_i : V \to V_i$$

$$\varphi = \sum_i P_i(\varphi) \qquad \forall \varphi \in V$$

ArbiLoMod Overview



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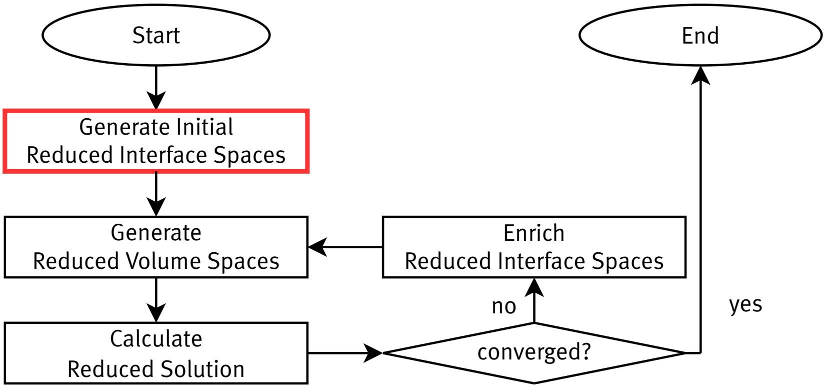


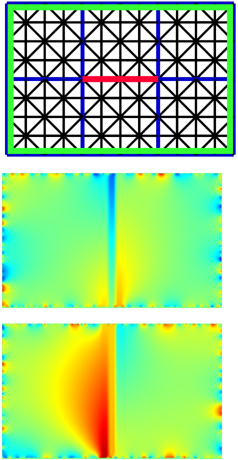
1. Generate Initial Interface Spaces

Adaption of “Pairwise Training” from PR-SCRBE[Eftang and Patera, 2014], Sixpack-Training:

On patch around interface:

- solve for random boundary conditions
- apply projection operator for interface
- construct space approximating all projected solutions

Similar in GMsFEM context:
“Randomized Oversampling” [Calo et al., 2014].



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Similar in GMsFEM context:
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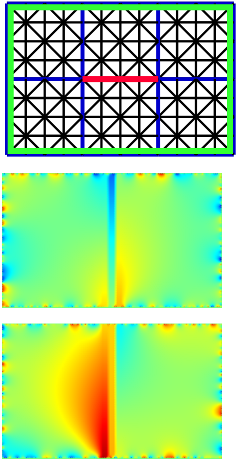
Only geometry data on six domain needed.



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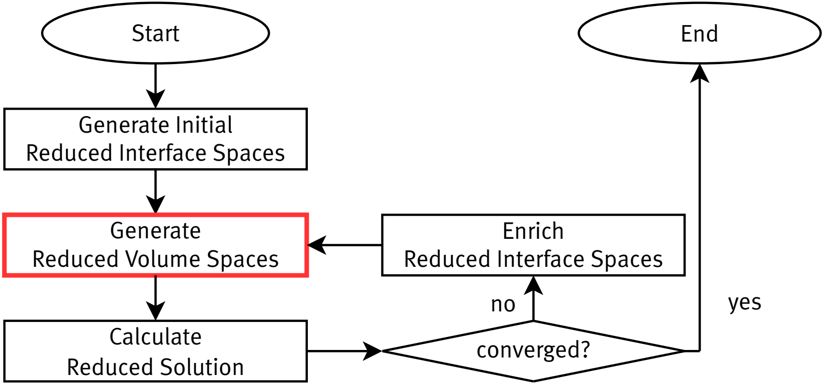

2. Generate Reduced Volume Spaces

Create volume basis which can approximate solution:
... for all interface functions
... for all parameter values in training set

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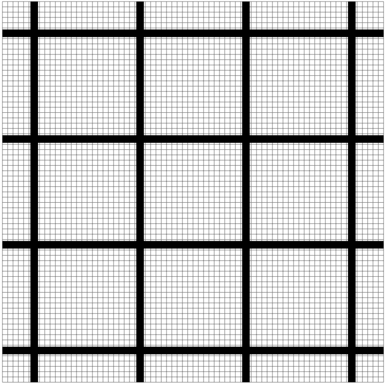

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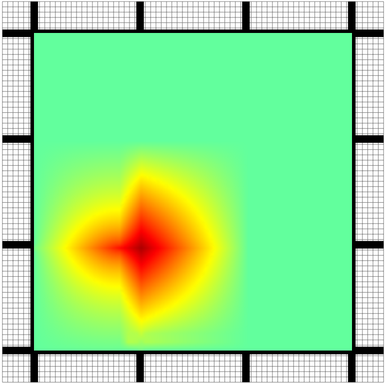

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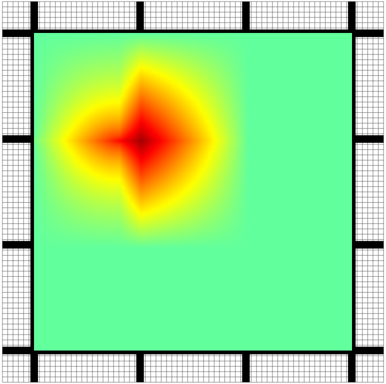

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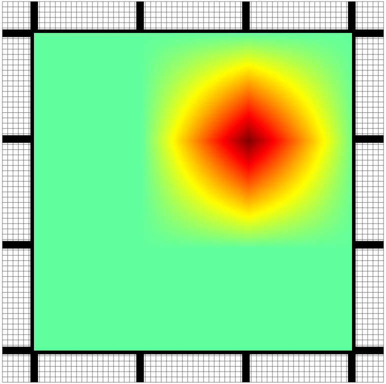

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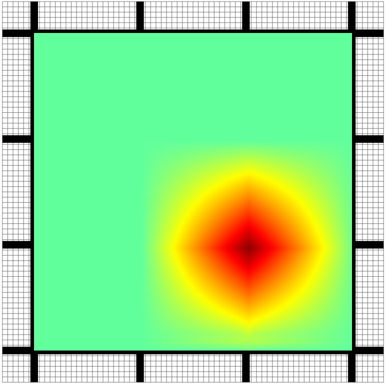

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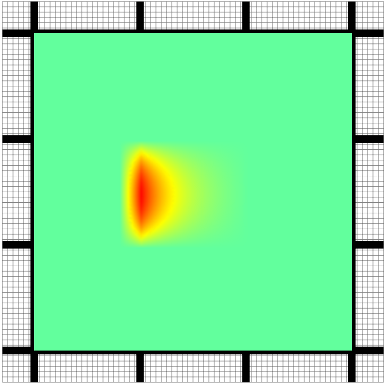

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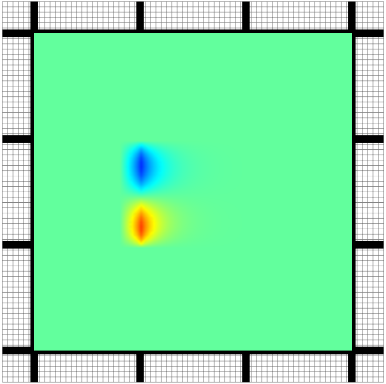

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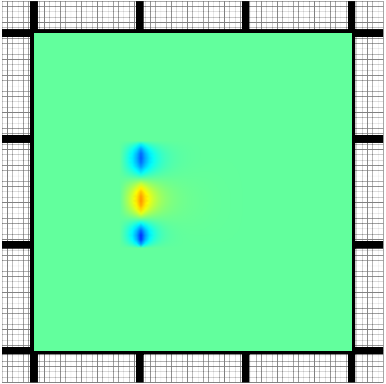

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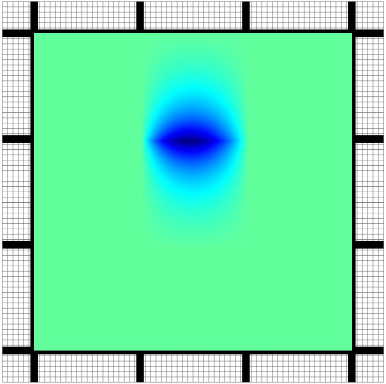

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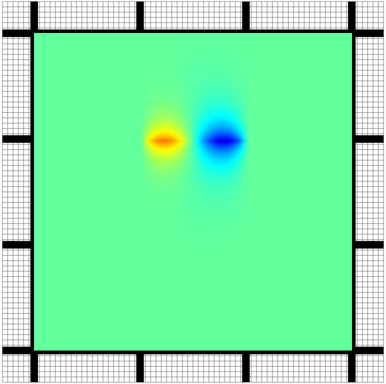

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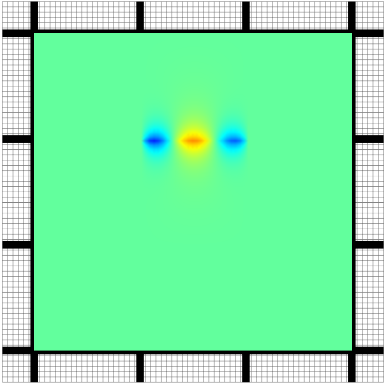

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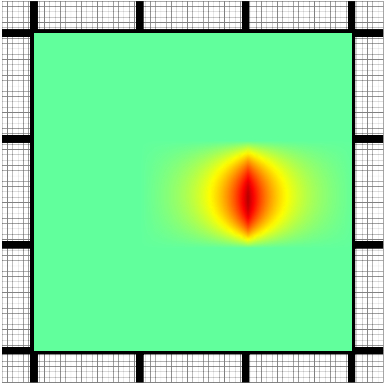

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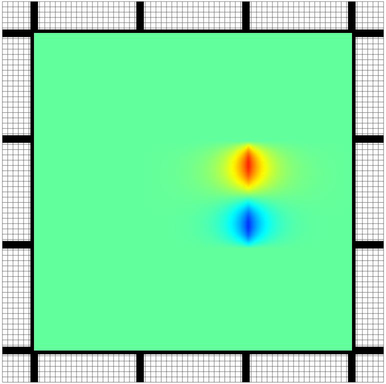

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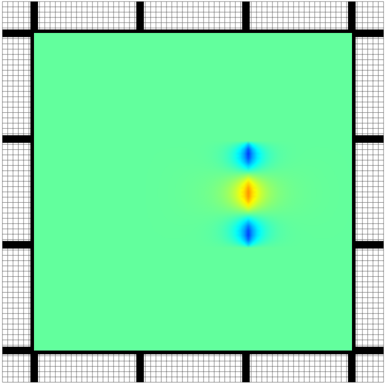

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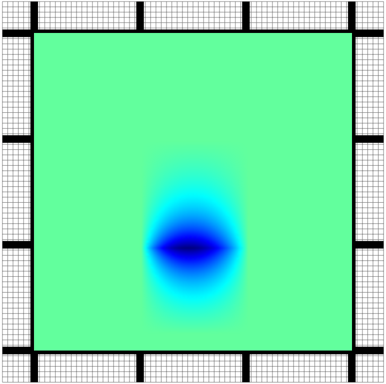

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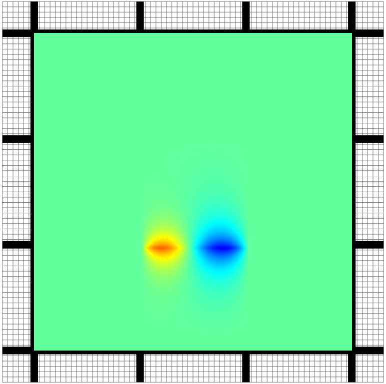

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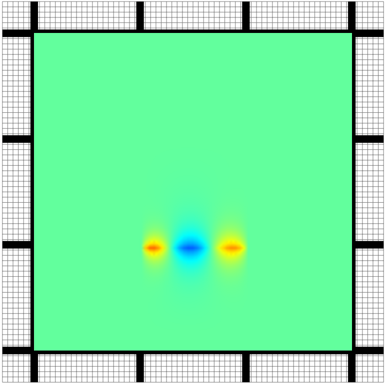

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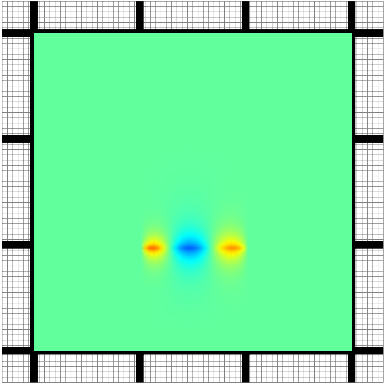

- Create space using local greedy.
- Done similar by Iapichino / Quarteroni [Iapichino et al., 2014].

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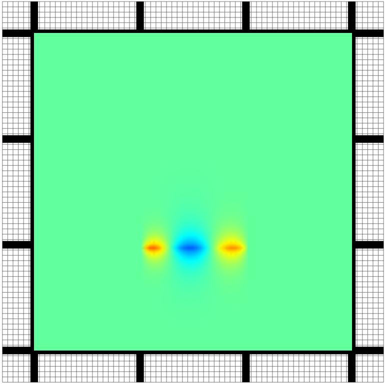

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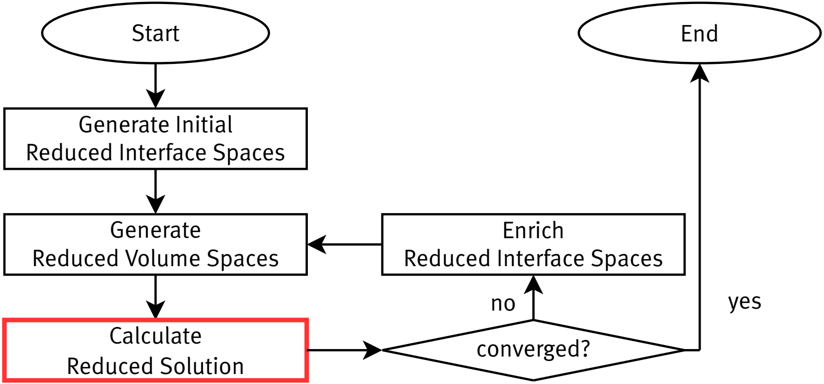
- Done similar by Iapichino / Quarteroni [Iapichino et al., 2014].

Only geometry data on nine domains needed.

ArbiLoMod Overview



3. Calculate Reduced Solution

Reduced Problem

Find $\widetilde{u}_\mu$ in $\widetilde{V}_{\text{LRB}}$ such that

$$a(\widetilde{u}_\mu, \widetilde{v}; \mu) = f(\widetilde{v}; \mu) \qquad \forall\, \widetilde{v} \in \widetilde{V}_{\text{LRB}}$$

Global Reduced Space

$$\widetilde{V}_{\text{LRB}} := \bigoplus_i \widetilde{V}_i$$

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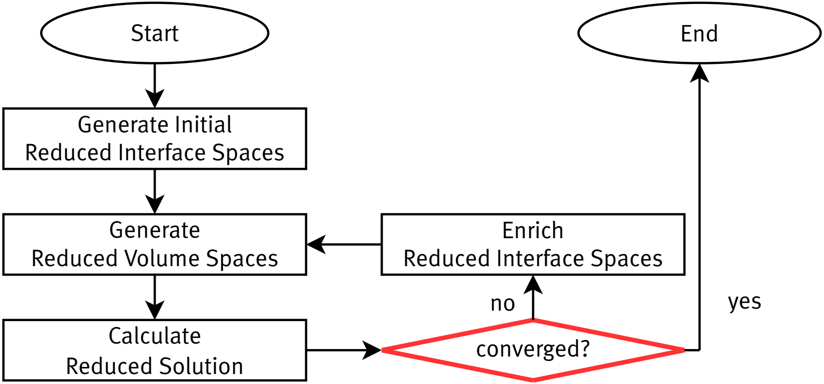
Global Reduced Space

$$\widetilde{V}_{\mathrm{LRB}} := \bigoplus_i \widetilde{V}_i$$

- Local assembly
- Local basis generation
- Local projection
  $\rightarrow$ only reduced quantities are communicated.

ArbiLoMod Overview



4. Localized A-Posteriori Error Estimator

Standard reduced basis error estimator:

$$||u_\mu - \widetilde{u}_\mu||_V \leq \frac{1}{\alpha_{LB}} ||\mathrm{R}_\mu(\widetilde{u}_\mu)||_{V'}.$$

u_μ	solution in V
$\widetilde{u}_\mu$	solution in $\widetilde{V}_{\mathrm{LRB}}$
α_{LB}	lower bound for coercivity constant
$\mathrm{R}_\mu(\widetilde{u}_\mu)$	residual

Localization of A-Posteriori Error Estimator

Standard reduced basis error estimator:

$$||u_\mu - \widetilde{u}_\mu||_V \leq \frac{1}{\alpha_{LB}} ||\mathrm{R}_\mu(\widetilde{u}_\mu)||_{V'}.$$

With:

$$V = \sum_i O_i \qquad O_i \subset V$$

$$P_{O_i} : V \rightarrow O_i$$

$$\varphi = \sum_i P_{O_i}(\varphi) \qquad \forall \varphi \in V$$

Localization

$$||\mathrm{R}_\mu(\widetilde{u}_\mu)||_{V'} \leq c_p \left(\sum_i ||\mathrm{R}_\mu(\widetilde{u}_\mu)||^2_{O_i'} \right)^{\frac{1}{2}} \qquad c_p := \max_{\varphi \in V \setminus \{0\}} \frac{\left(\sum_i ||P_{O_i}(\varphi)||^2_V\right)^{\frac{1}{2}}}{||\varphi||_V}$$

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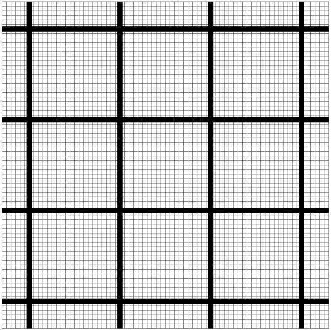
Space Decomposition

$$O_i \subset V \qquad V = \sum_i O_i$$

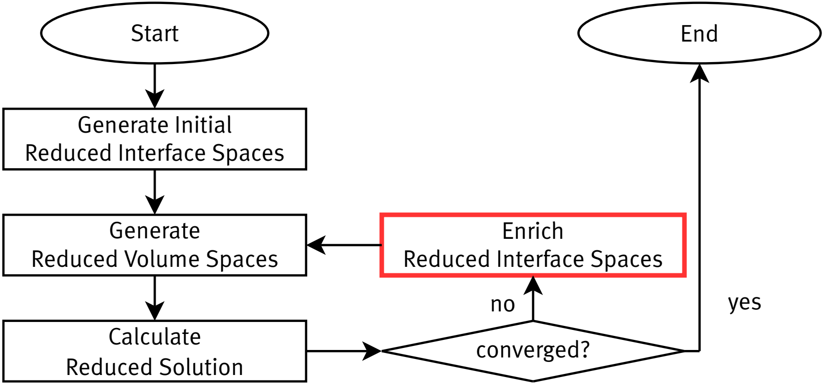
$$||u_\mu - \tilde{u}_\mu||_V \leq \frac{c_p}{\alpha_{LB}} \sum_i \left(||\mathrm{R}_\mu(\tilde{u}_\mu)||^2_{O_i'} \right)^{\frac{1}{2}}$$

$$c_p := \max_{\varphi \in V \setminus \{0\}} \frac{\left(\sum_i ||P_{O_i}(\varphi)||^2_V\right)^{\frac{1}{2}}}{||\varphi||_V}$$

- O_k here: inner dofs of overlapping patches
- P_{O_k} here: multiplication with partition of unity



ArbiLoMod Overview



Enrichment

1. Find largest residual:

$$\hat{k}, \hat{\mu} \leftarrow \underset{\mu,k}{\arg\max} \, ||R_\mu(\widetilde{u}_\mu)||_{O'_k}$$

2. Solve local problem:
Find $u_l \in O_{\hat{k}}$ such that:

$$a(u_l, \varphi; \hat{\mu}) = R_{\hat{\mu}}(\widetilde{u}_{\hat{\mu}})(\varphi) \qquad \forall \varphi \in O_{\hat{k}}$$

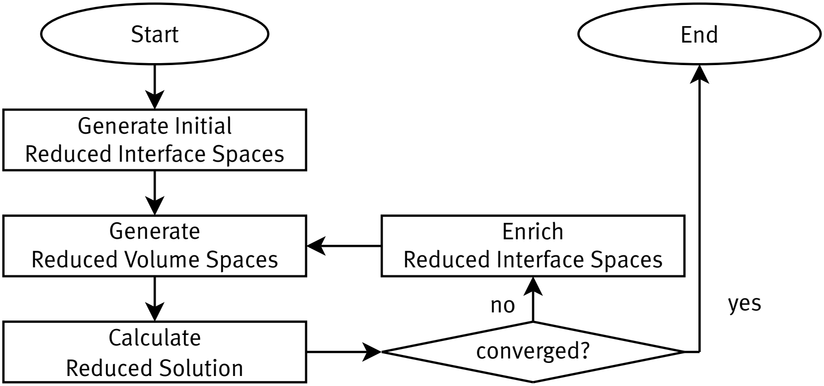
3. Apply space decomposition:

$$\hat{i} \leftarrow \underset{i}{\arg\max} \, || \left(P_{V_i}(u_l)\right)^\perp ||_{V_i}$$

$$\widetilde{V}_{\hat{i}} \leftarrow \widetilde{V}_{\hat{i}} \oplus \operatorname{span}\left(\left(P_{V_i}(u_l)\right)^\perp\right)$$

With $(P_{V_i}(u_l))^\perp$: part of $P_{V_i}(u_l)$ orthogonal to $\widetilde{V}_i$.
Similar to online adaptivity in GMsFEM [Chung et al., 2015].

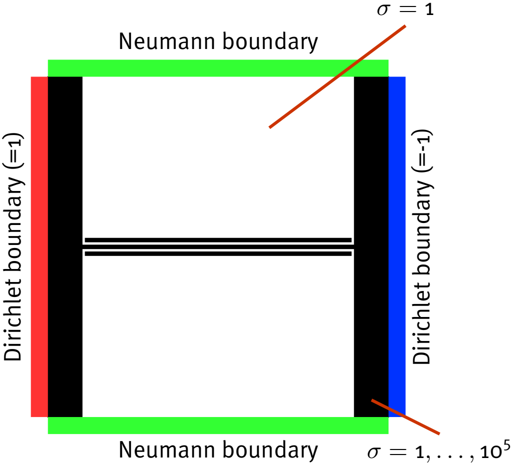
ArbiLoMod Overview



Numerical Example

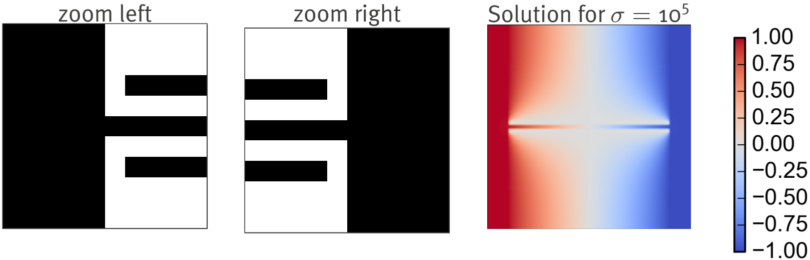
Heat conduction
$-\nabla \cdot (\sigma \nabla u) = 0$

Implemented in pyMOR[Milk et al., 2015].



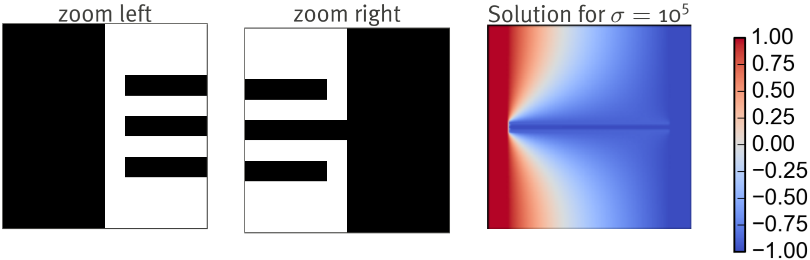
Sequence of Geometries

Geometry 1:



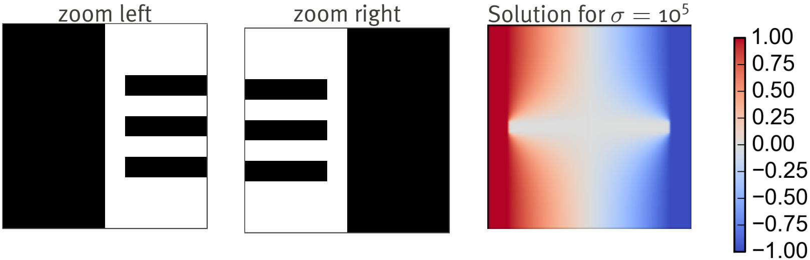
Sequence of Geometries

Geometry 2:



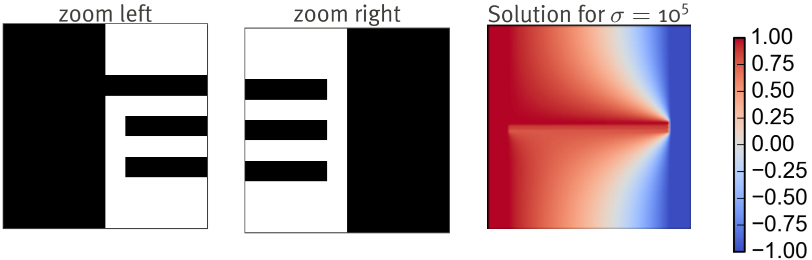
Sequence of Geometries

Geometry 3:



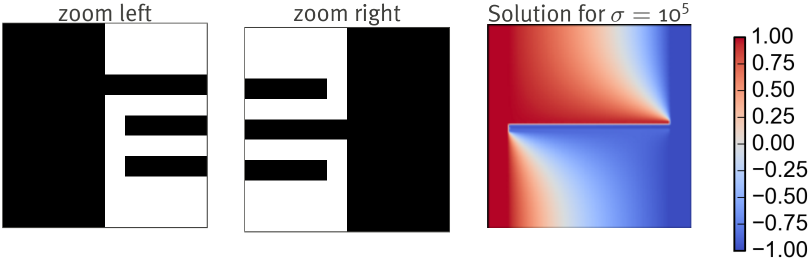
Sequence of Geometries

Geometry 4:



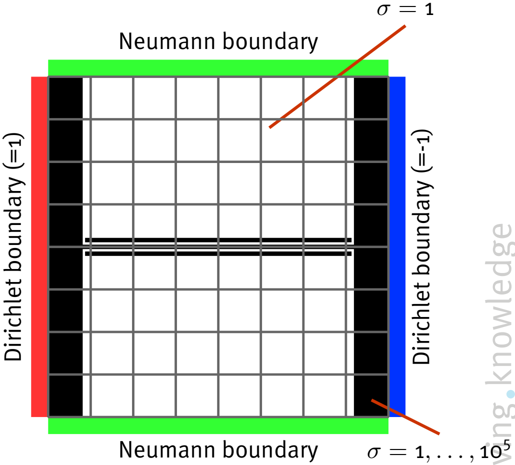
Sequence of Geometries

Geometry 5:

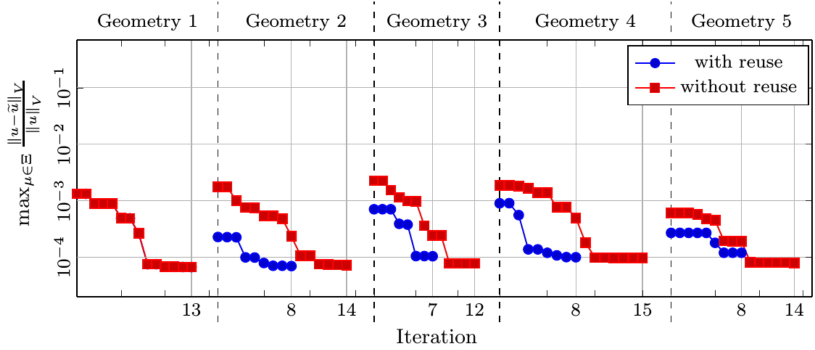


Numerical Example

- Heat conduction
  $-\nabla \cdot (\sigma \nabla u) = 0$
- 8x8 domain decomposition
- 80.401 dofs in full model



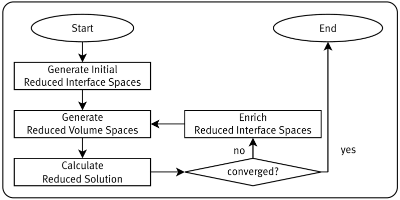
Convergence



Savings

geometry	trainings		greedys		iterations	
	reuse:		reuse:		reuse:	
	no	yes	no	yes	no	yes
1	112	112	64	64	14	14
2	112	5 (-96 %)	64	8 (-88 %)	15	9 (-40 %)
3	112	5 (-96 %)	64	8 (-88 %)	13	8 (-38 %)
4	112	3 (-97 %)	64	6 (-91 %)	16	9 (-44 %)
5	112	5 (-96 %)	64	8 (-88 %)	15	9 (-40 %)

Summary



ArbiLoMod

- ... handles arbitrary local modifications.
- ... intended for interactive use.
- ... is based on the Reduced Basis Method.
- ... has localized and adaptive generation of basis vectors.
- ... is communication avoiding.

Acknowledgements

Many thanks to...

CST - Computer Simulation Technology AG[1]

... for sponsoring my research.

[1] www.cst.com

References

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