



WESTFÄLISCHE  
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MÜNSTER

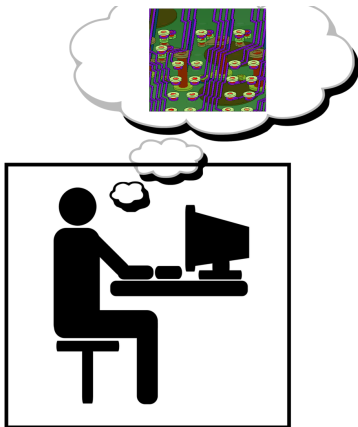


APPLIED  
MATHEMATICS  
MÜNSTER

# Cloud Based Interactive Simulations of Maxwell's Equations Using the Localized Reduced Basis Method

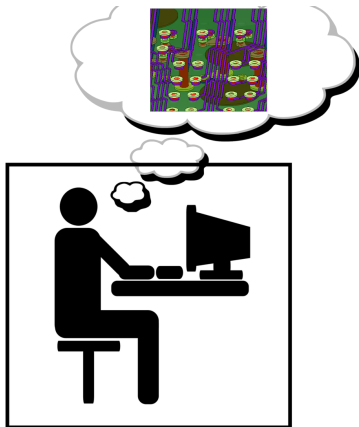
ICIAM 2015 - Beijing

# Non-Parametric Geometry Changes



Envision engineer working on design

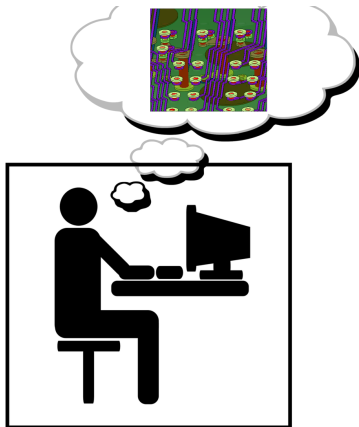
## Non-Parametric Geometry Changes



Envision engineer working on design

Multi-query setting:  
Sequence of geometries

# Non-Parametric Geometry Changes



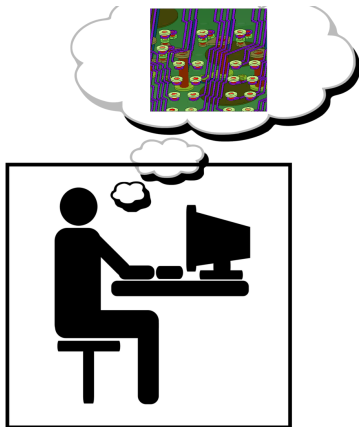
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Multi-query setting:  
Sequence of geometries

Properties of changes:

1. very localized

## Non-Parametric Geometry Changes



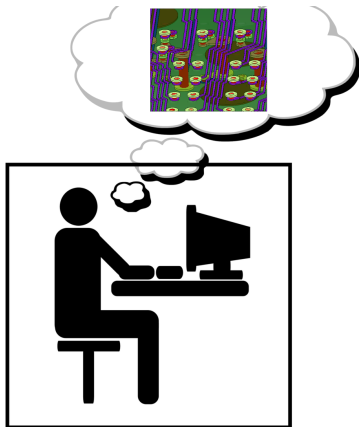
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## Non-Parametric Geometry Changes



Envision engineer working on design

Multi-query setting:  
Sequence of geometries

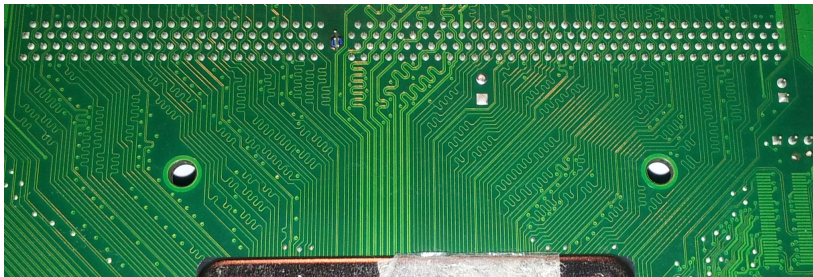
Properties of changes:

1. very localized
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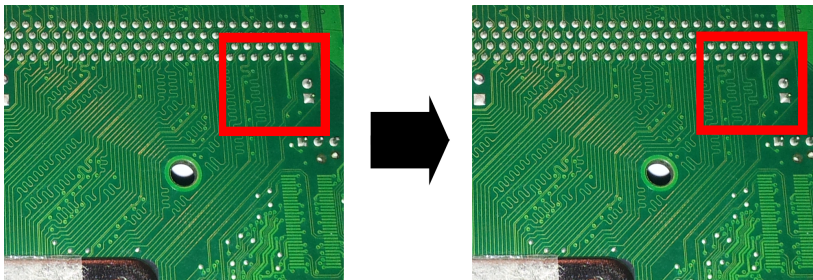
Cluster often available.  
Cloud always available.



## Example

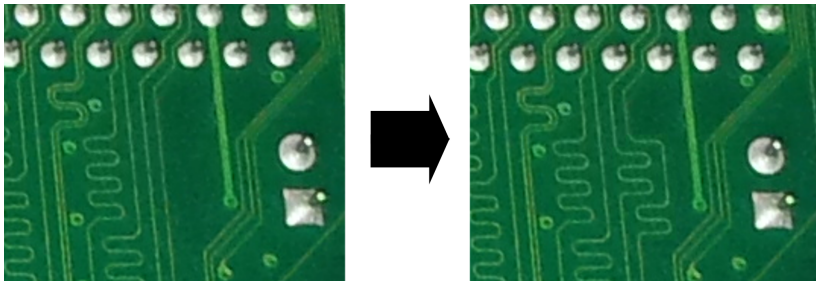


## Example

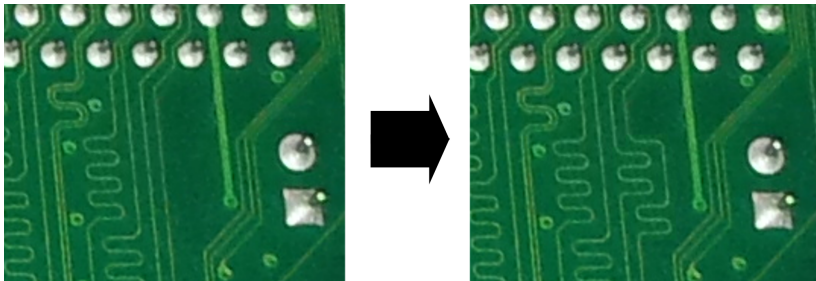




## Example



## Example



Goal: time harmonic Maxwell's equations  
Examples here: heat conduction



## Main Idea

- ▶ Modify the Reduced Basis Method (RBM) to use basis functions with local support.
- ▶ After geometry change, reuse basis functions which do not have support in changed area.

## Setting

- ▶ Coercive, continuous, parametric bilinear form

$$a_\mu : V \times V \rightarrow \mathbb{R}$$

$$\alpha_{LB} \|u\|_V \leq a_\mu(u, u)$$

- ▶ Linear, parametric form

$$f_\mu : V \rightarrow \mathbb{R}$$

- ▶ Problem: find  $u_\mu$  in  $V$  such that

$$a_\mu(u_\mu, v) = f_\mu(v) \quad \forall v \in V$$



## Standard Reduced Basis

### Problem

Find  $u_\mu$  in  $V$  such that

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### Subspace Construction

Construct subspace  $\tilde{V} \subset V$  with  $\dim(\tilde{V}) \ll \dim(V)$ ,

$$\tilde{V} := \text{span}\{u_{\mu_1}, u_{\mu_2}, u_{\mu_3}, \dots\}$$

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Find  $\tilde{u}_\mu$  in  $\tilde{V}$  such that

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## Standard Reduced Basis

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## Localized Reduced Basis

Space decomposition into local subspaces

$$V = \bigoplus_i V_i \quad V_i \subset V$$

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Find reduced local spaces

$$\tilde{V}_i \subset V_i$$

Construct global reduced space

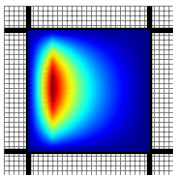
$$\tilde{V}_{\text{LRB}} := \bigoplus_i \tilde{V}_i$$

## Space Decomposition

- ▶ Nonoverlapping Domain Decomposition:

$$\Omega = \bigcup_i \Omega_i \quad \Omega_i \cap \Omega_j = \emptyset \quad i \neq j$$

- ▶  $V$ : First order Lagrange on mesh.



Codim-0 Space

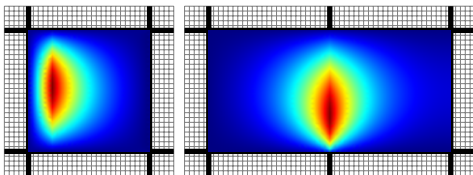
- mesh line
- domain boundary

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Codim-0 Space

Codim-1 Space

— mesh line

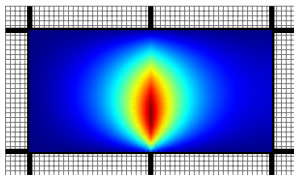
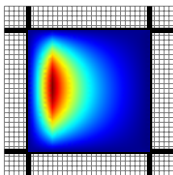
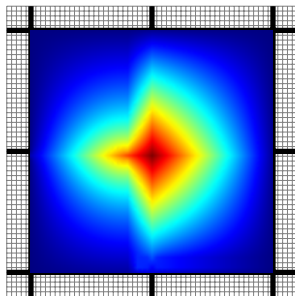
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Codim-0 Space

Codim-1 Space

Codim-2 Space

— mesh line

— domain boundary



# Projection Operators

- ▶ Direct decomposition of ansatz space:

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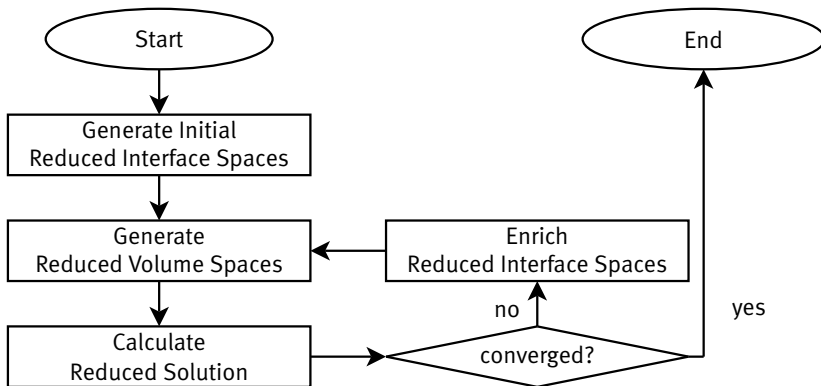
$$V = \bigoplus_i V_i \quad V_i \subset V$$

- ▶ Definition of projection operators:

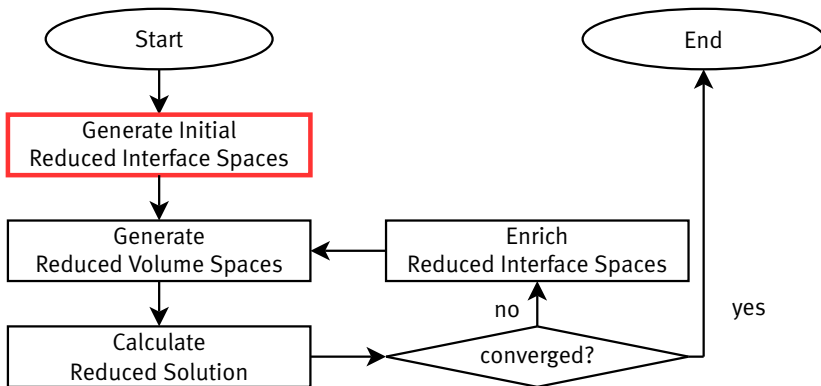
$$P_i : V \rightarrow V_i$$
$$\varphi = \sum_i P_i(\varphi) \quad \forall \varphi \in V$$



## ArbiLoMod Overview



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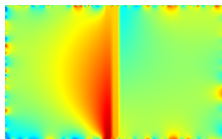
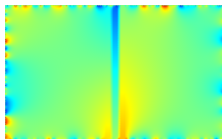
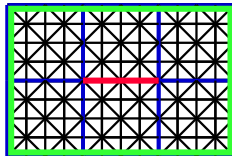
# 1. Generate Initial Interface Spaces

Adaption of “Pairwise Training” from  
PR-SCRBE[6],  
Sixpack-Training:

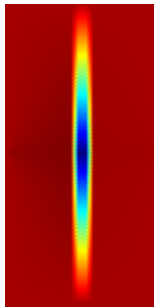
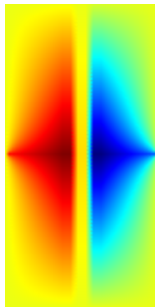
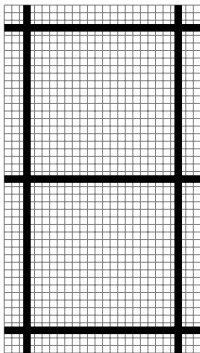
On patch around interface:

- ▶ solve for random boundary conditions
- ▶ apply projection operator for interface
- ▶ construct space approximating all projected solutions

Similar in GMSFEM context:  
“Randomized Oversampling” [3].

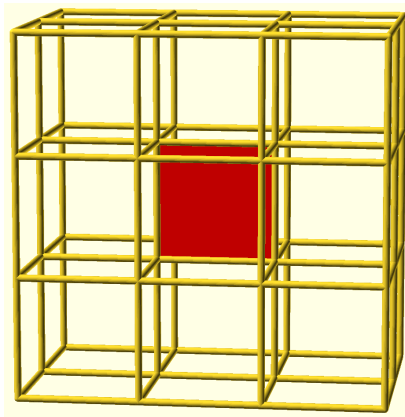
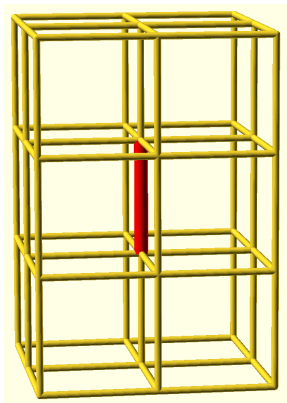


## Example Interface Functions

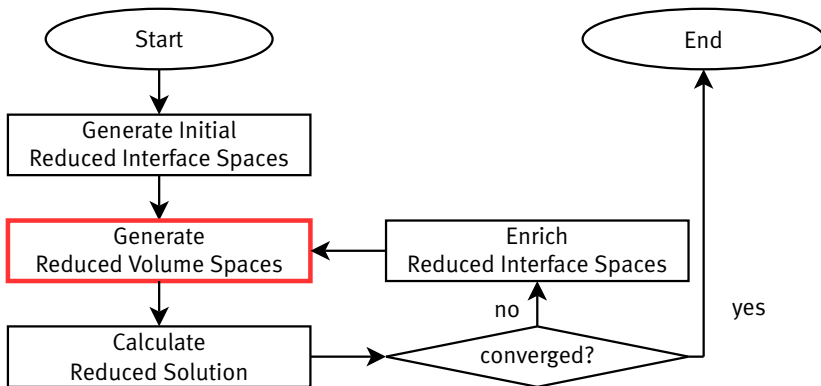


## Note on 3D

Training environments in 3D (not implemented):



## ArbiLoMod Overview





## 2. Generate Reduced Volume Spaces

Create volume basis which can approximate solution:

... for all interface functions

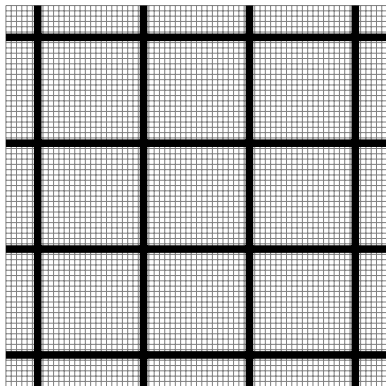
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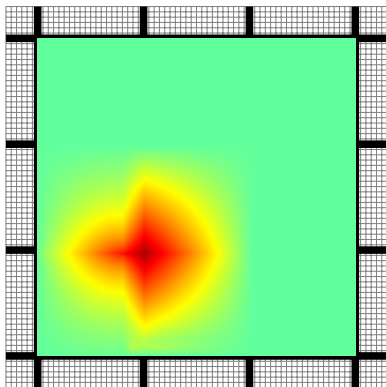


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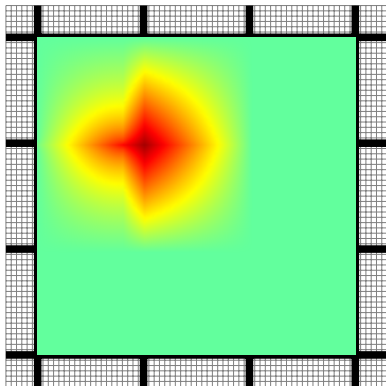


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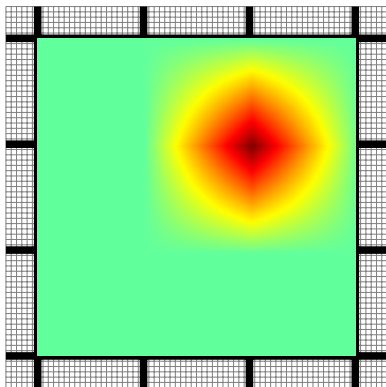


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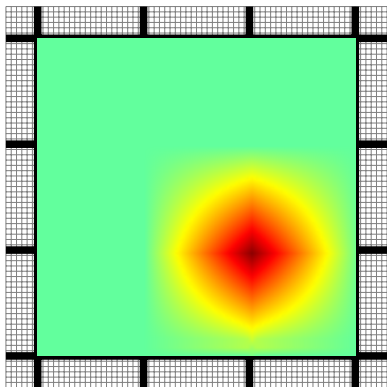


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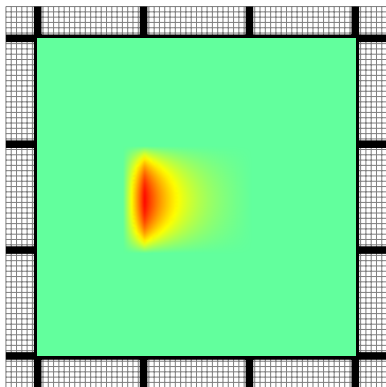


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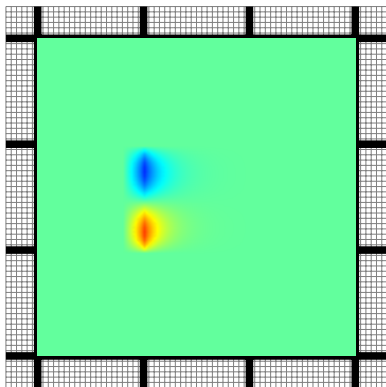


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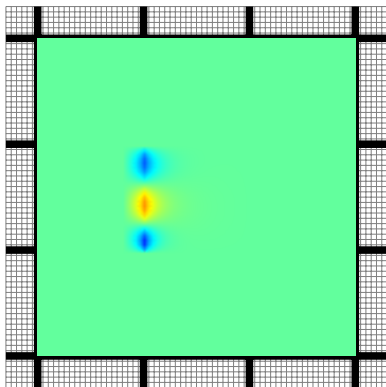


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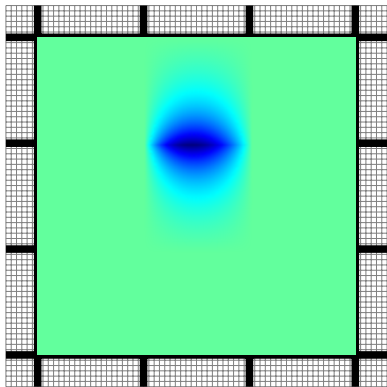


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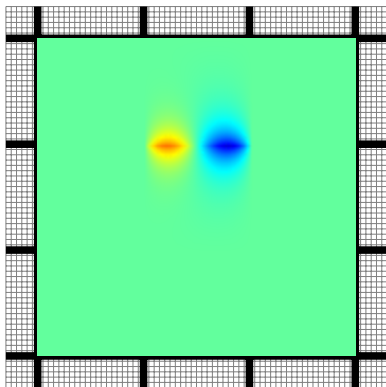


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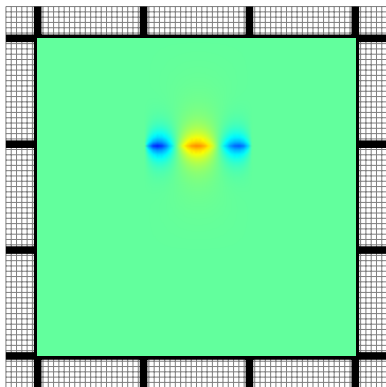


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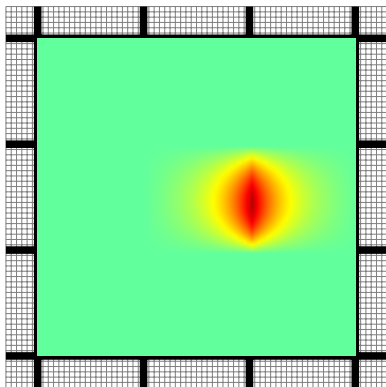


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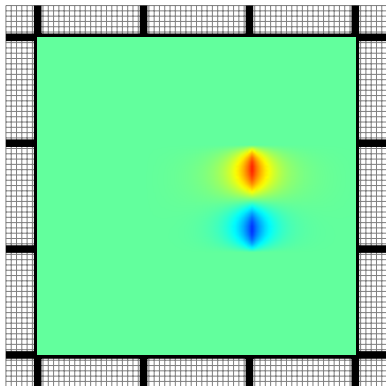


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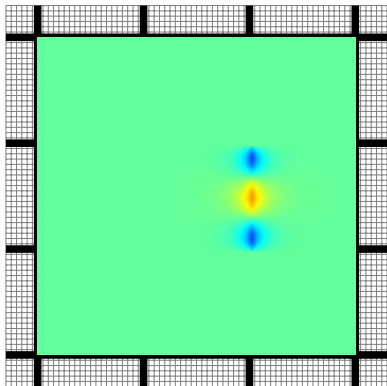


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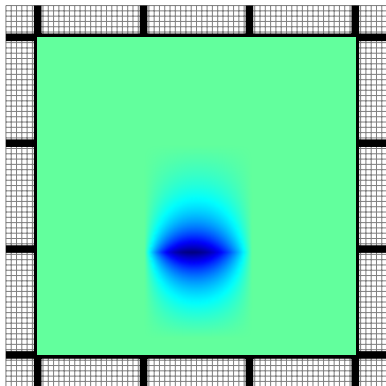


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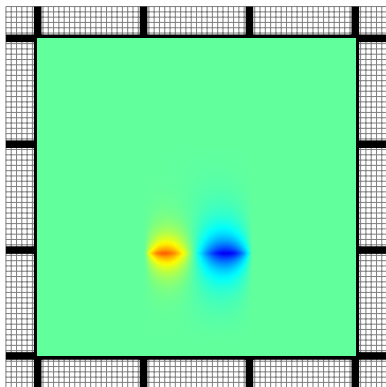


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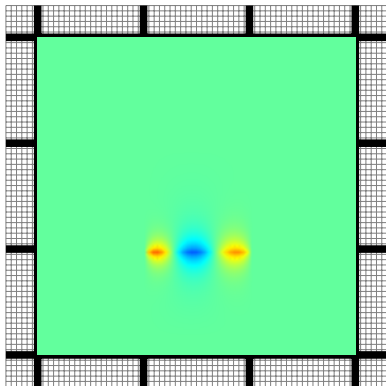


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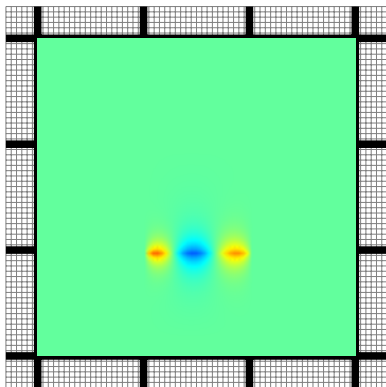


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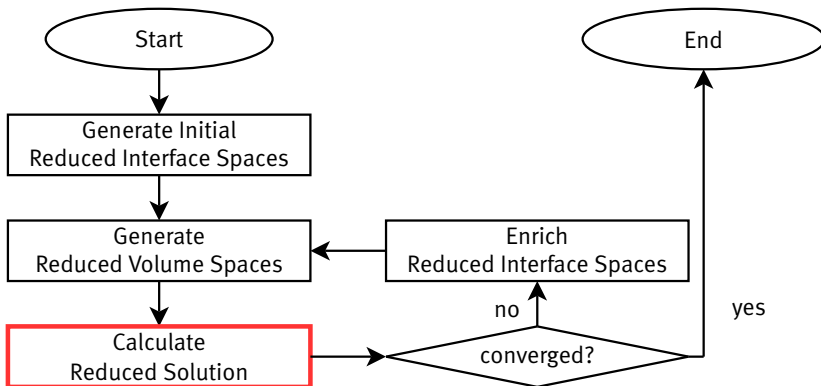
... for all interface functions

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- ▶ Create space using local greedy.
- ▶ Done similar by lapichino / Quarteroni [7].

## ArbiLoMod Overview



### 3. Calculate Reduced Solution

#### Reduced Problem

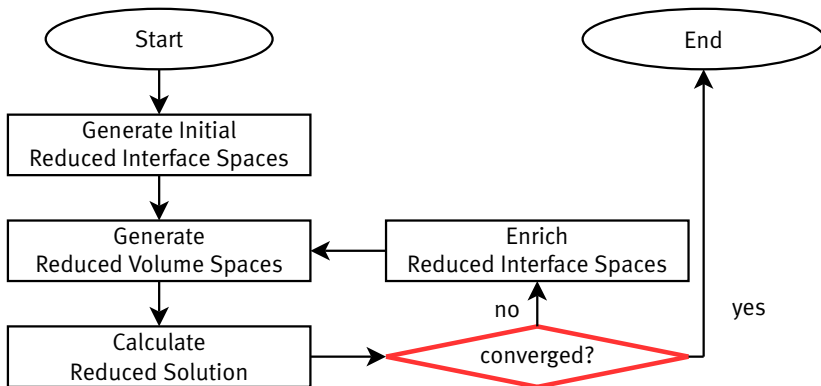
Find  $\tilde{u}_\mu$  in  $\tilde{V}_{\text{LRB}}$  such that

$$a_\mu(\tilde{u}_\mu, \tilde{v}) = f_\mu(\tilde{v}) \quad \forall \tilde{v} \in \tilde{V}_{\text{LRB}}$$

#### Global Reduced Space

$$\tilde{V}_{\text{LRB}} := \bigoplus_i \tilde{V}_i$$

## ArbiLoMod Overview



## 4. Localized A-Posteriori Error Estimator

Standard reduced basis error estimator:

$$\|u_\mu - \tilde{u}_\mu\|_V \leq \frac{1}{\alpha_{LB}} \|R_\mu(\tilde{u}_\mu)\|_{V'}.$$

## 4. Localized A-Posteriori Error Estimator

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$u_\mu$  solution in  $V$

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$\alpha_{LB}$  lower bound for coercivity constant



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$u_\mu$  solution in  $V$

$\tilde{u}_\mu$  solution in  $\tilde{V}_{LRB}$

$\alpha_{LB}$  lower bound for coercivity constant

$R_\mu(\tilde{u}_\mu)$  residual

## Localization of A-Posteriori Error Estimator

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With:

$$V = \sum_i V_i \quad V_i \subset V$$

$$P_i : V \rightarrow V_i$$

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### Localization

$$\|R_\mu(\tilde{u}_\mu)\|_{V'} \leq \sum c_i \|R_\mu(\tilde{u}_\mu)\|_{V'_i} \quad c_i := \max_{\varphi \in V} \frac{\|P_i(\varphi)\|_V}{\|\varphi\|_V}$$



## Space Decomposition

- ▶ Different space decomposition for a-posteriori error estimator.

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$$\|u_\mu - \tilde{u}_\mu\|_V \leq \frac{1}{\alpha_{LB}} \sum c_i \|R_\mu(\tilde{u}_\mu)\|_{O'_i}$$

$$c_i := \max_{\varphi \in V} \frac{\|P_{O_k}(\varphi)\|_V}{\|\varphi\|_V}$$

- ▶  $O_k$  here: inner dofs of overlapping patches
- ▶  $P_{O_k}$  here: multiplication with partition of unity

## Space Decomposition

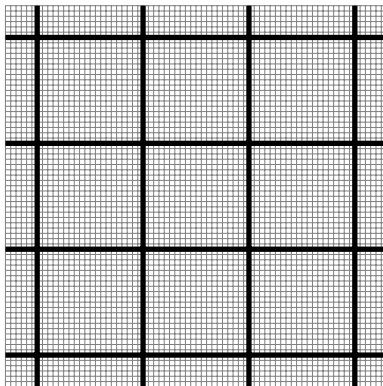
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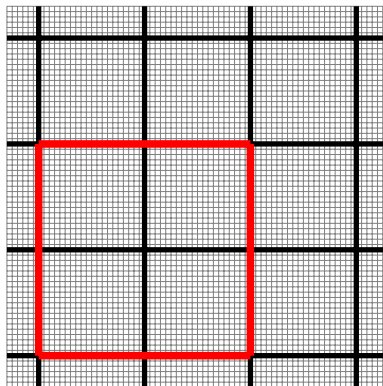
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## Space Decomposition

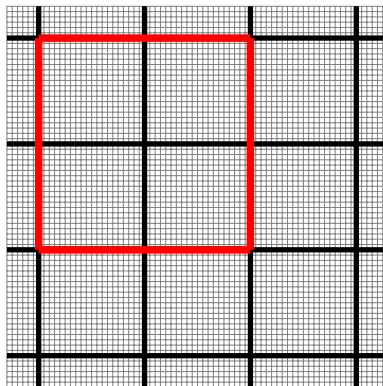
- ▶ Different space decomposition for a-posteriori error estimator.

$$O_k \subset V \quad V = \sum_k O_k$$

$$\|u_\mu - \tilde{u}_\mu\|_V \leq \frac{1}{\alpha_{LB}} \sum c_i \|R_\mu(\tilde{u}_\mu)\|_{O_i'}$$

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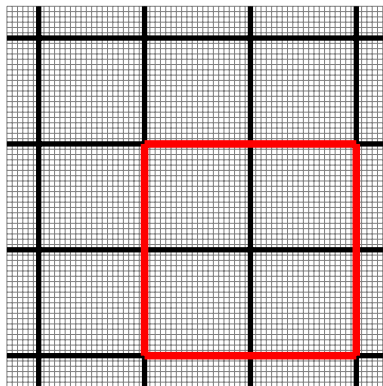
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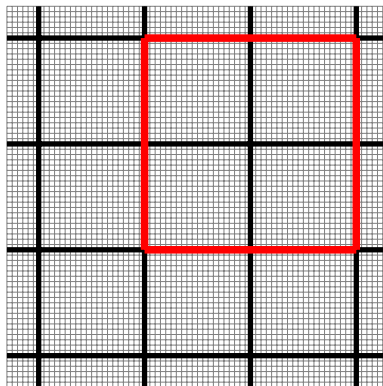
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## A-Posteriori Error Estimator

$$\|u_\mu - \tilde{u}_\mu\|_V \leq \frac{1}{\alpha_{LB}} \sum c_i \|R_\mu(\tilde{u}_\mu)\|_{O'_i}$$

- ▶ It is rigorous and efficient.
- ▶ It is online-offline decomposable.
- ▶ It is parallelizable with little amount of communication.
- ▶ After a localized geometry change, the offline computed data in unaffected regions can be reused.

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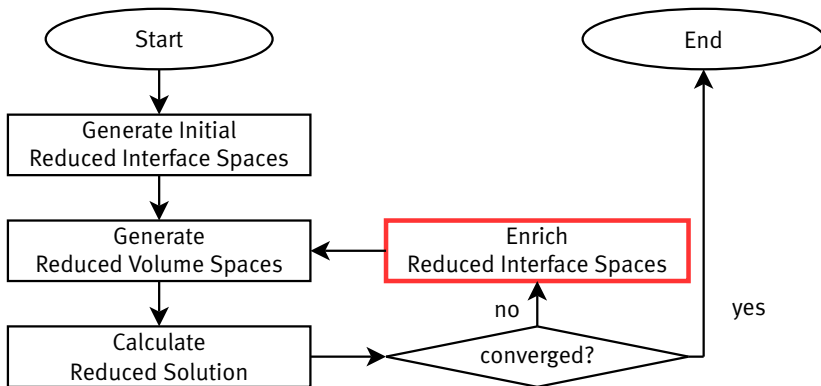
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Local residuals on overlapping spaces are also used in GMsFEM.

## ArbiLoMod Overview





## 5. Enrich Interface Spaces

Two main questions:

- ▶ Which space to enrich?



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- ▶ Which space to enrich?
- ▶ How to enrich?





## Which Space to Enrich?

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Space decomposition into local subspaces

$$V = \bigoplus_i V_i \quad V_i \subset V$$

Find reduced local spaces

$$\tilde{V}_i \subset V_i$$

Construct global reduced space

$$\tilde{V}_{\text{LRB}} := \bigoplus_i \tilde{V}_i$$



## Which Space to Enrich?

Enrich  $V_{i_{\max}}$  using  $\mu_{\max}$  where

$$(i_{\max}, \mu_{\max}) := \arg \max_{i, \mu} \|R_{\mu}(\tilde{u}_{\mu})\|_{V_i'}$$

... does not work!



## Enrichment

1. Find largest residual:

$$\hat{k}, \hat{\mu} \leftarrow \arg \max_{\mu, k} \|R_{\mu}(\tilde{u}_{\mu})\|_{\sigma'_k}$$

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3. Apply space decomposition:

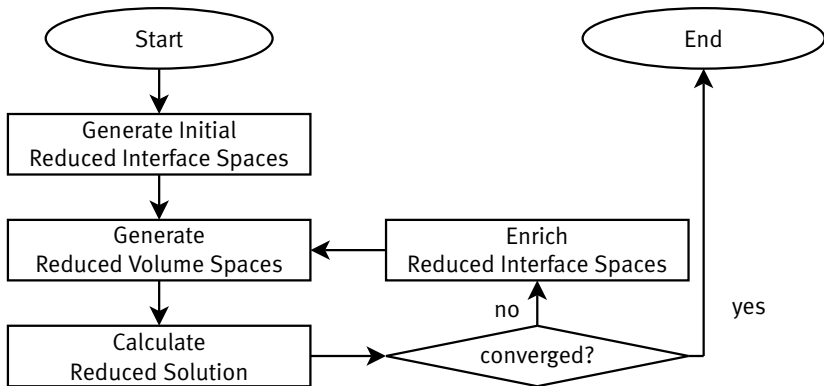
$$\hat{i} \leftarrow \arg \max_i \| (P_{V_i}(u_l))^{\perp} \|_{V_i}$$

$$\tilde{V}_{\hat{i}} \leftarrow \tilde{V}_{\hat{i}} \oplus \text{span} \left( (P_{V_i}(u_l))^{\perp} \right)$$

With  $(P_{V_i}(u_l))^{\perp}$  : part of  $P_{V_i}(u_l)$  orthogonal to  $\tilde{V}_i$ .

Similar to online adaptivity in GMsFEM [4].

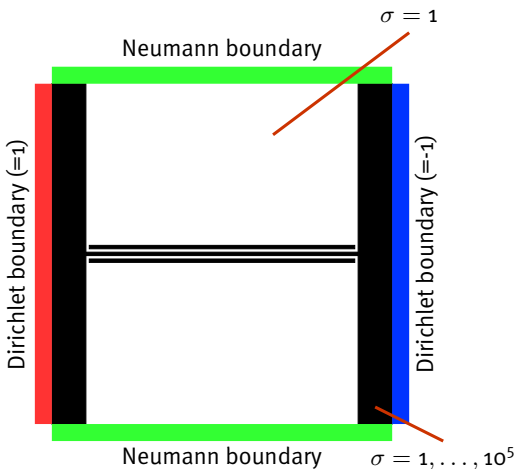
## ArbiLoMod Overview



## Numerical Example

Heat conduction

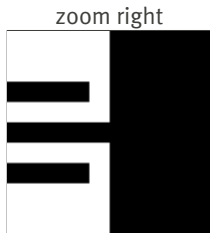
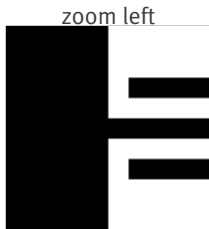
$$-\nabla \cdot (\sigma \nabla u) = 0$$



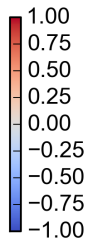
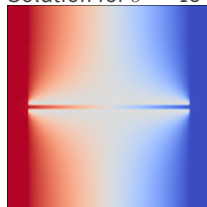


## Sequence of Geometries

Geometry 1:

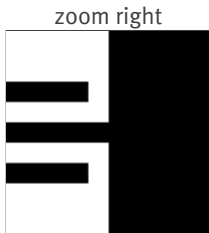
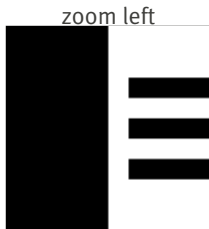


Solution for  $\sigma = 10^5$

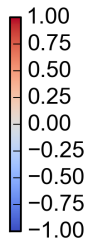
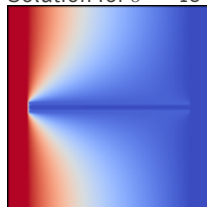


## Sequence of Geometries

Geometry 2:

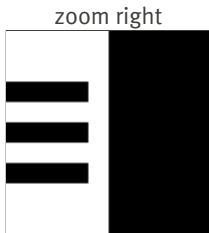
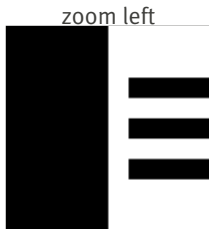


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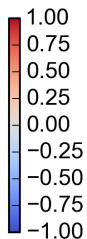
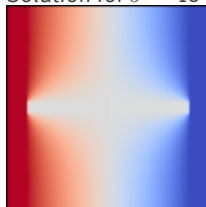


## Sequence of Geometries

Geometry 3:

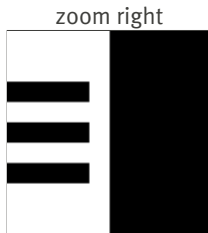
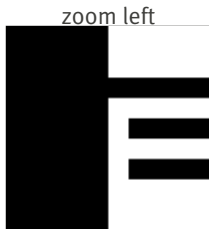


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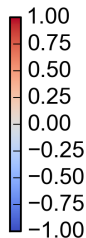
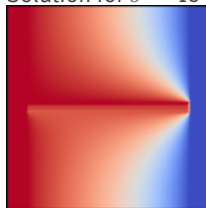


## Sequence of Geometries

Geometry 4:

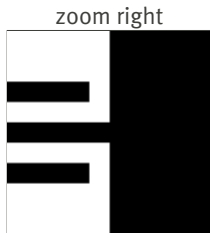
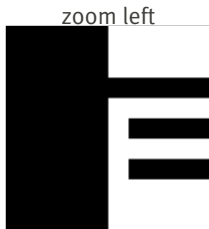


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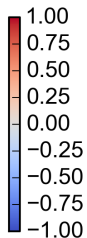
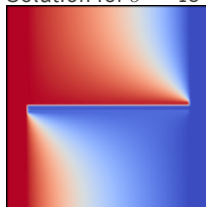


## Sequence of Geometries

Geometry 5:

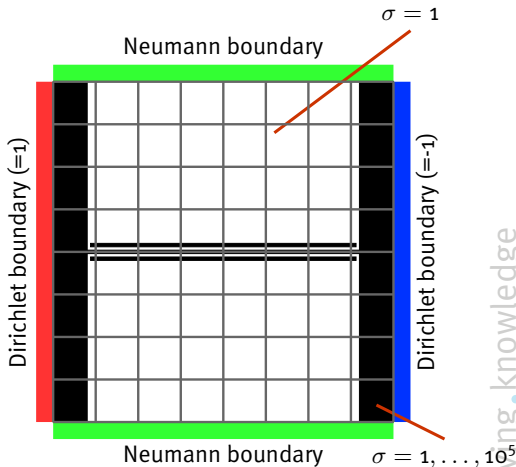


Solution for  $\sigma = 10^5$

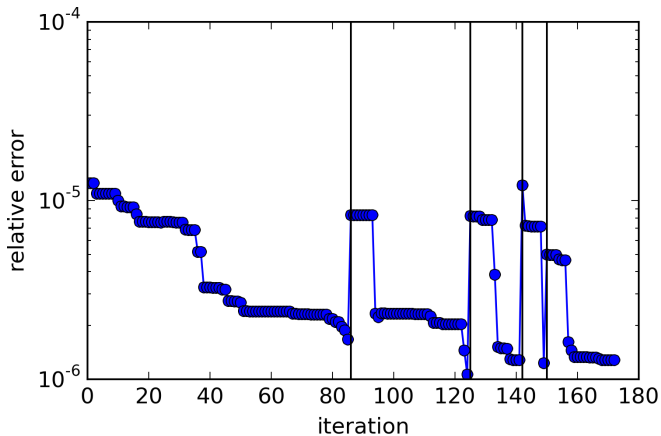


## Numerical Example

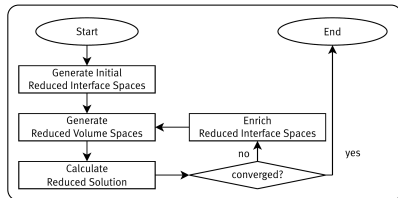
- ▶ Heat conduction  
 $-\nabla \cdot (\sigma \nabla u) = 0$
- ▶ 8x8 domain decomposition
- ▶ 80.401 dofs in full model



## Convergence



## Summary



### ArbiLoMod

- ▶ ... handles arbitrary local modifications.
- ▶ ... intended for interactive use.
- ▶ ... is based on the Reduced Basis Method.
- ▶ ... has localized and adaptive generation of basis vectors.



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## Acknowledgements

Many thanks to...

CST - Computer Simulation Technology AG<sup>1</sup>

... for sponsoring my research.

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<sup>1</sup>[www.cst.com](http://www.cst.com)