Westfälische Wilhelms-Universität Münster
Applied Mathematics Münster

ArbiLoMod: a Localized Reduced Basis Approach to Handle Arbitrary Local Modifications

ENUMATH 2015 - Ankara

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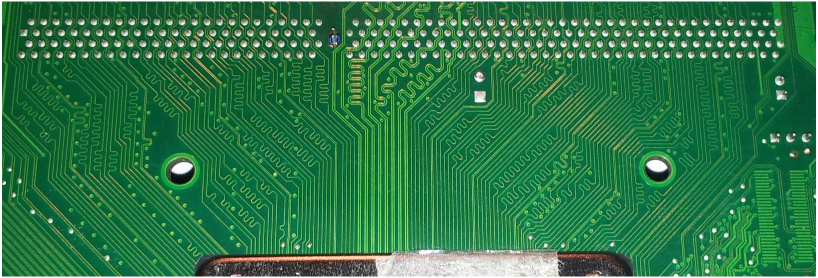
September 17, 2015

Example



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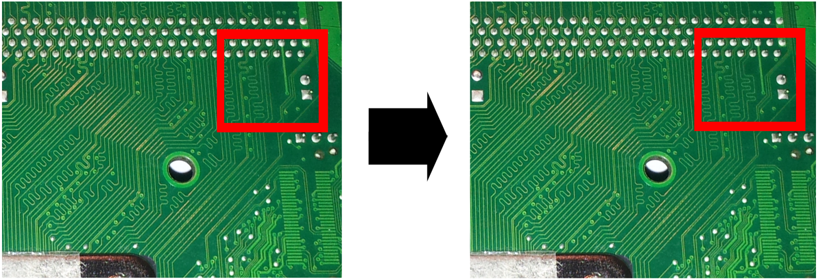

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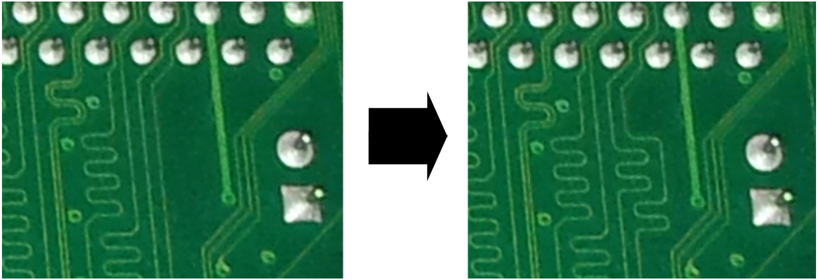
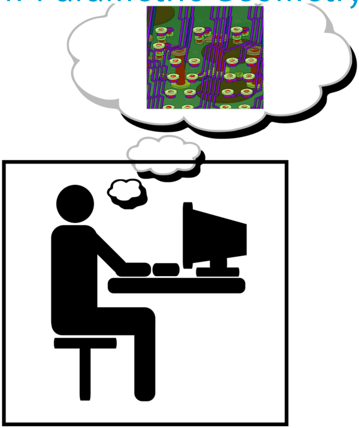


Non-Parametric Geometry Changes



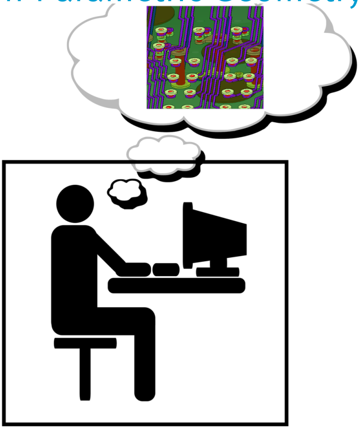
Multi-query setting:
Sequence of geometries

Properties of changes:

1. very localized
2. unforeseen

Cluster often available.
Cloud always available.

Non-Parametric Geometry Changes



Multi-query setting:
Sequence of geometries

Properties of changes:

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Cluster often available.
Cloud always available.

Goal: time harmonic Maxwell's equations
Examples here: heat conduction

Setting

- Coercive, continuous, parametric bilinear form

$$a : V \times V \times \mathcal{P} \to \mathbb{R}$$

$$\alpha_{LB} ||u||_V^2 \leq a(u, u, \mu) \qquad \forall \mu \in \mathcal{P}$$

- Linear, parametric form

$$f : V \times \mathcal{P} \to \mathbb{R}$$

- Problem: find u_μ in V such that

$$a(u_\mu, v; \mu) = f(v; \mu) \qquad \forall v \in V$$

Main Idea

- Modify the Reduced Basis Method (RBM) to use basis functions with local support.
- After geometry change, reuse basis functions which do not have support in changed area.

ArbiLoMod Ingredients

1. Localized spaces

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1. Localized spaces
2. Localized basis generation (training and greedy)
3. Localized a-posteriori error estimation
4. Localized adaptive basis enrichment

Standard Reduced Basis

Problem

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Subspace Construction

Construct subspace $\widetilde{V} \subset V$ with $\dim(\widetilde{V}) \ll \dim(V)$,

$$\widetilde{V} := \mathrm{span}\{u_{\mu_1}, u_{\mu_2}, u_{\mu_3}, \dots\}$$

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Reduced Problem

Find $\widetilde{u}_\mu$ in $\widetilde{V}$ such that

$$a(\widetilde{u}_\mu, \widetilde{v}; \mu) = f(\widetilde{v}; \mu) \qquad \forall\, \widetilde{v} \in \widetilde{V}$$

Standard Reduced Basis

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Find u_μ in V such that

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Subspace Construction - To Be Changed

Construct subspace $\widetilde{V} \subset V$ with $\dim(\widetilde{V}) \ll \dim(V)$,

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Reduced Problem

Find $\widetilde{u}_\mu$ in $\widetilde{V}$ such that

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Localized Reduced Basis

Space decomposition into local subspaces

$$V = \bigoplus_i V_i \qquad V_i \subset V$$

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$$\widetilde{V}_i \subset V_i$$

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Space decomposition into local subspaces

$$V = \bigoplus_i V_i \qquad V_i \subset V$$

Find reduced local spaces

$$\widetilde{V}_i \subset V_i$$

Construct global reduced space

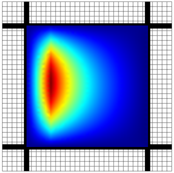
$$\widetilde{V}_{\text{LRB}} := \bigoplus_i \widetilde{V}_i$$

Space Decomposition

- Nonoverlapping Domain Decomposition:

$$\Omega = \bigcup_i \Omega_i \qquad \Omega_i \cap \Omega_j = \emptyset \quad i \neq j$$

- V: First order Lagrange on mesh.



Codim-0 Space

mesh line
domain boundary

Space Decomposition

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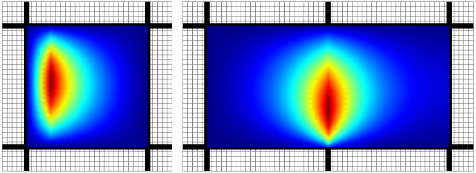

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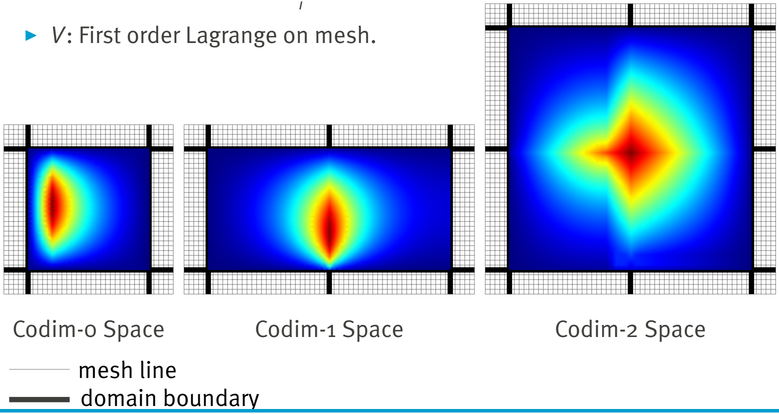

Projection Operators

- Direct decomposition of ansatz space:

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- Definition of projection operators:

$$P_i : V \to V_i$$

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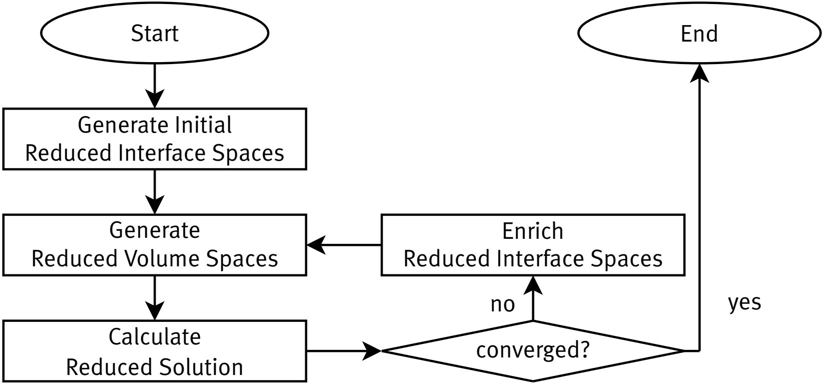
ArbiLoMod Overview



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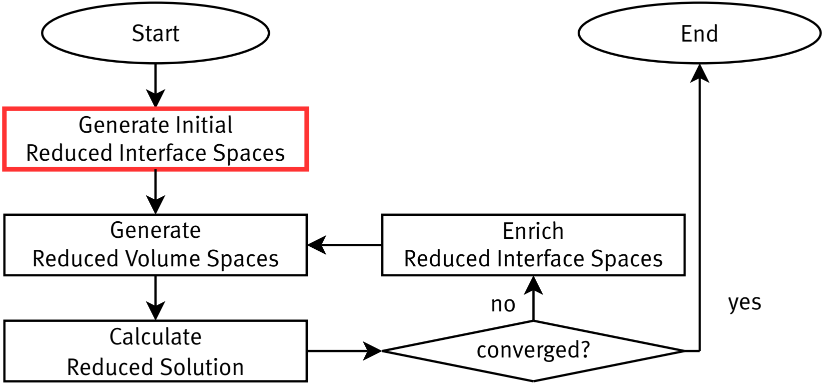


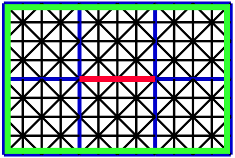
1. Generate Initial Interface Spaces

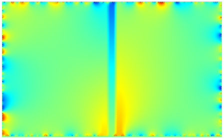
Adaption of “Pairwise Training” from PR-SCRBE[Eftang and Patera, 2014], Sixpack-Training:

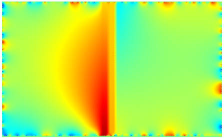
On patch around interface:

- solve for random boundary conditions
- apply projection operator for interface
- construct space approximating all projected solutions

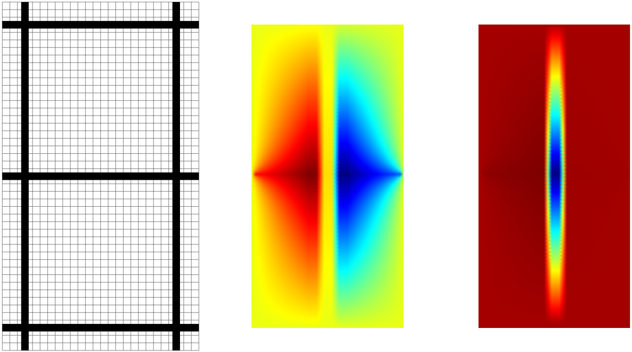
Similar in GMsFEM context: “Randomized Oversampling” [Calo et al., 2014].







Example Interface Functions



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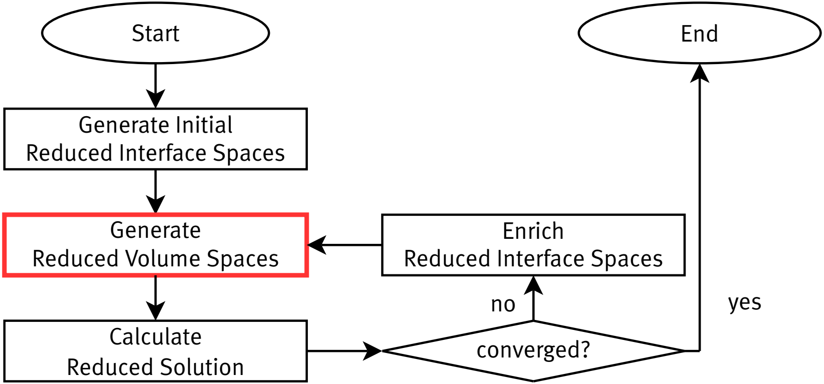

2. Generate Reduced Volume Spaces

Create volume basis which can approximate solution:
... for all interface functions
... for all parameter values in training set

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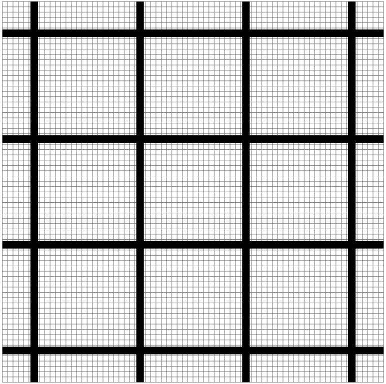

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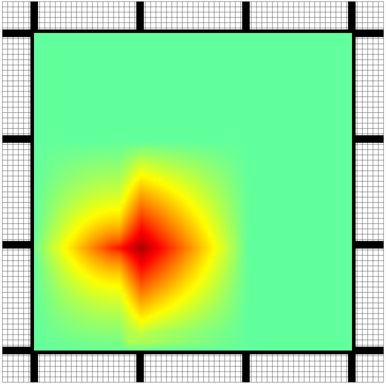

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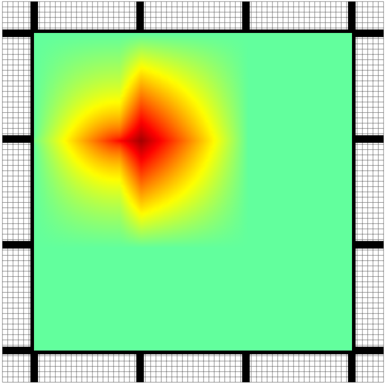

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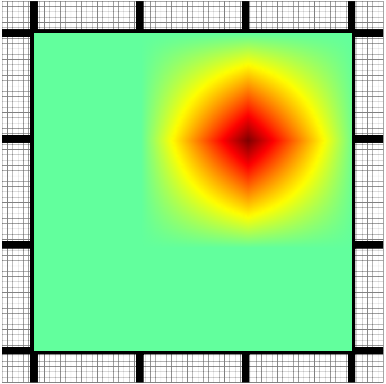

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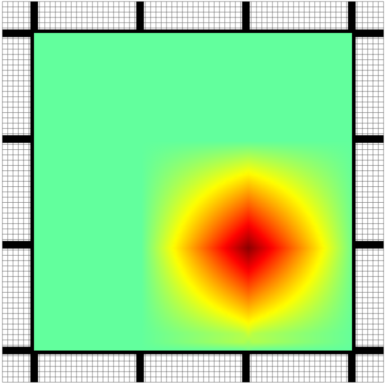

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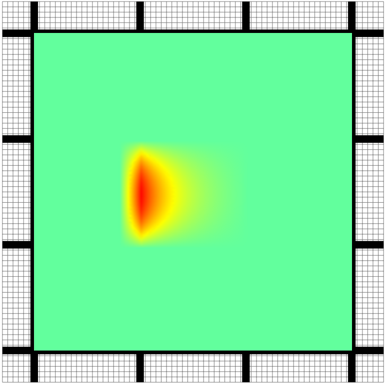

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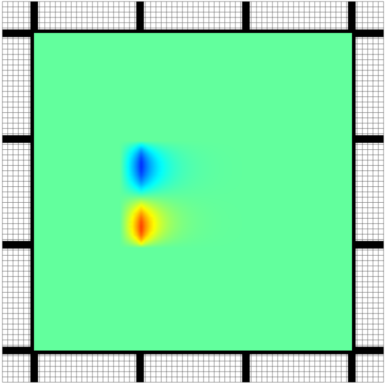

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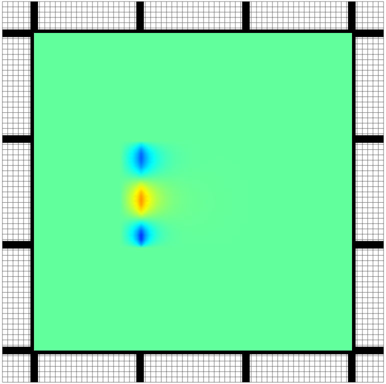

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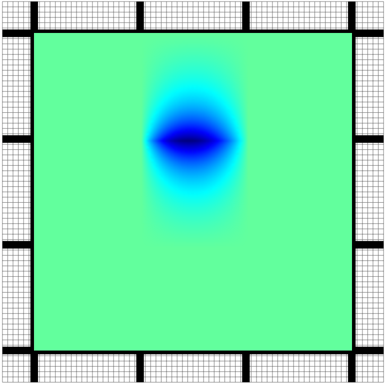

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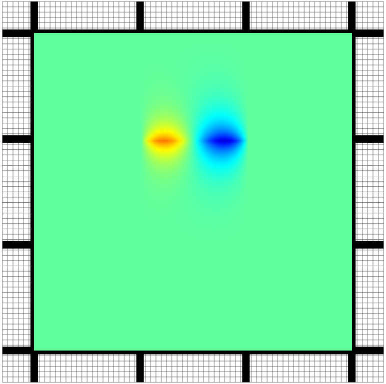

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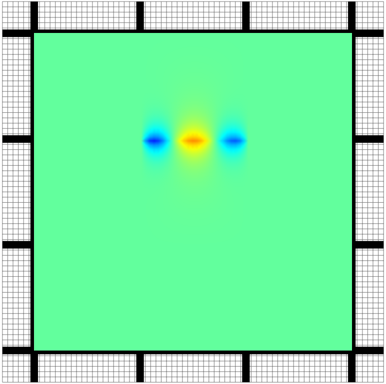

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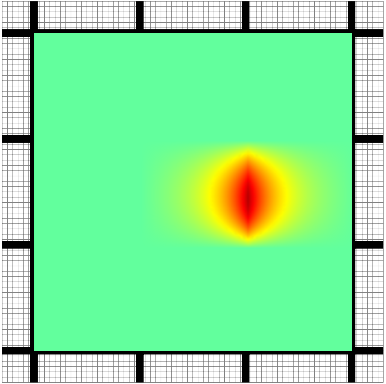

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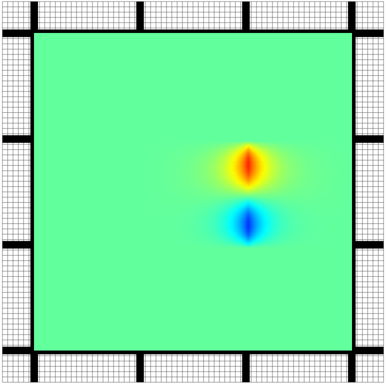

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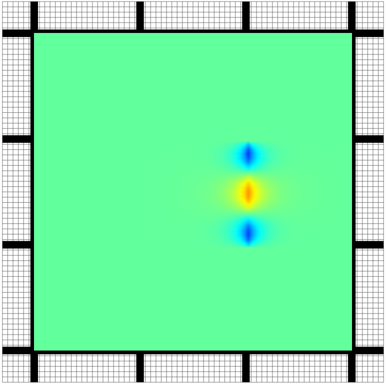

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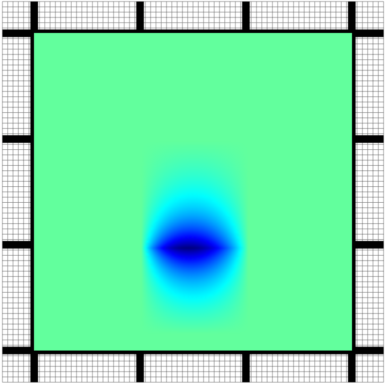

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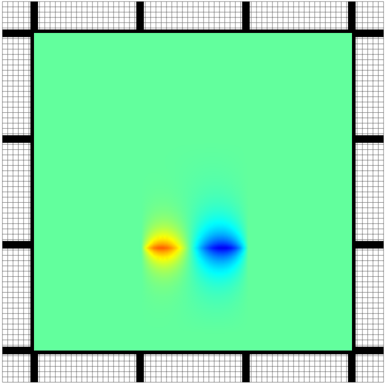

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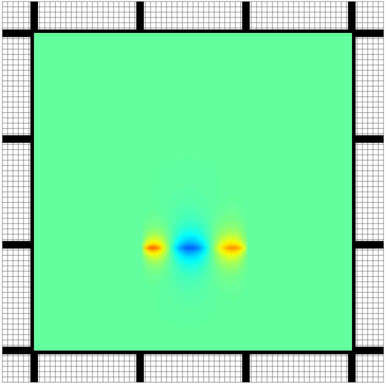

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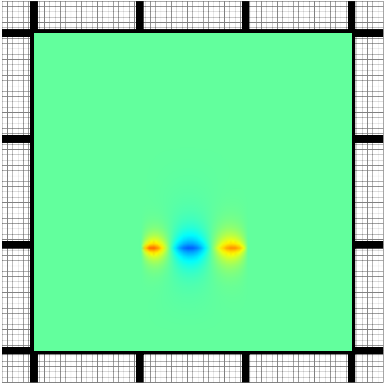
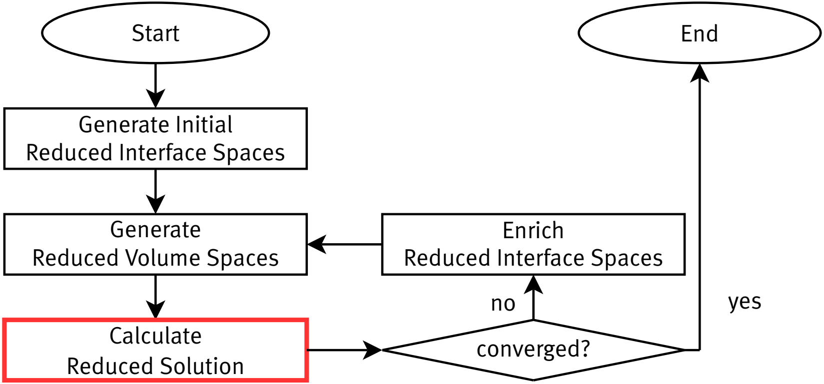


- Create space using local greedy.
- Done similar by Iapichino / Quarteroni [Iapichino et al., 2014].

ArbiLoMod Overview



3. Calculate Reduced Solution

Reduced Problem

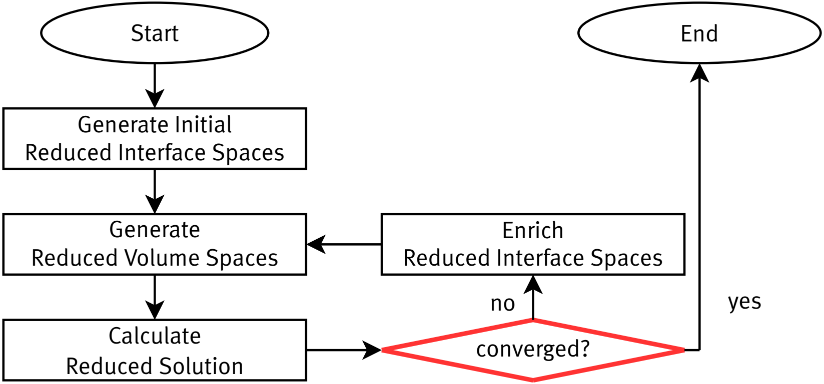
Find $\widetilde{u}_\mu$ in $\widetilde{V}_{\text{LRB}}$ such that

$$a(\widetilde{u}_\mu, \widetilde{v}; \mu) = f(\widetilde{v}; \mu) \qquad \forall\, \widetilde{v} \in \widetilde{V}_{\text{LRB}}$$

Global Reduced Space

$$\widetilde{V}_{\text{LRB}} := \bigoplus_i \widetilde{V}_i$$

ArbiLoMod Overview



4. Localized A-Posteriori Error Estimator

Standard reduced basis error estimator:

$$||u_\mu - \widetilde{u}_\mu||_V \leq \frac{1}{\alpha_{LB}} ||\mathrm{R}_\mu(\widetilde{u}_\mu)||_{V'}.$$

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u_μ solution in V

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α_{LB}	lower bound for coercivity constant
$\mathrm{R}_\mu(\widetilde{u}_\mu)$	residual

Localization of A-Posteriori Error Estimator

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With:

$$V = \sum_i V_i \qquad V_i \subset V$$

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Localization

$$||\mathrm{R}_\mu(\widetilde{u}_\mu)||_{V'} \leq \sum c_i ||\mathrm{R}_\mu(\widetilde{u}_\mu)||_{V_i'} \qquad c_i := \max_{\varphi \in V} \frac{||P_i(\varphi)||_V}{||\varphi||_V}$$

Space Decomposition

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$$c_i := \max_{\varphi \in V} \frac{||P_{O_k}(\varphi)||_V}{||\varphi||_V}$$

- O_k here: inner dofs of overlapping patches
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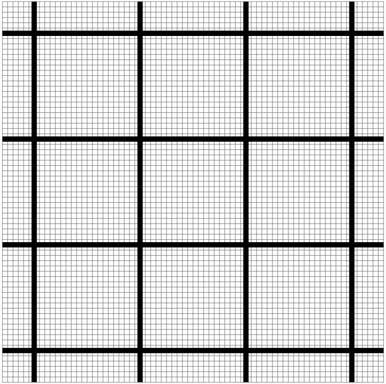

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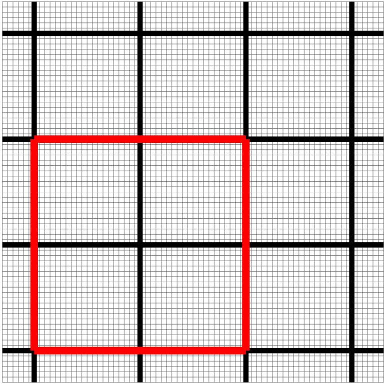

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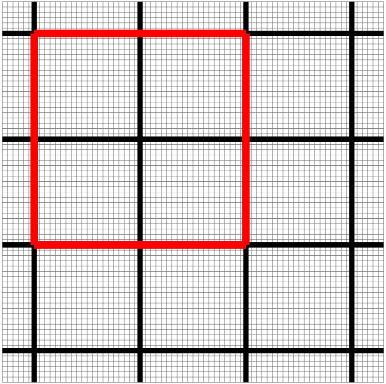

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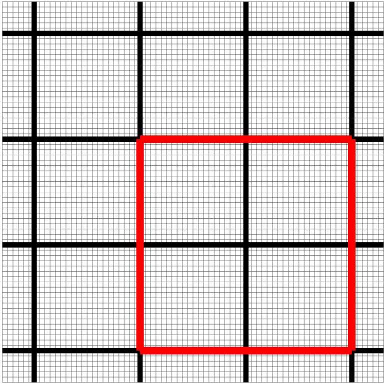
$$||u_\mu - \tilde{u}_\mu||_V \leq \frac{1}{\alpha_{LB}} \sum c_i ||R_\mu(\tilde{u}_\mu)||_{O_i'}$$

$$c_i := \max_{\varphi \in V} \frac{||P_{O_k}(\varphi)||_V}{||\varphi||_V}$$

- O_k here: inner dofs of overlapping patches
- P_{O_k} here: multiplication with partition of unity



Space Decomposition

- Different space decomposition for a-posteriori error estimator.

$$O_k \subset V \qquad V = \sum_k O_k$$

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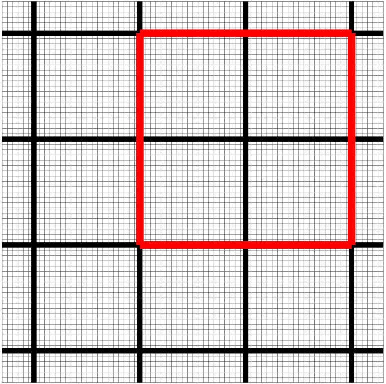


A-Posteriori Error Estimator

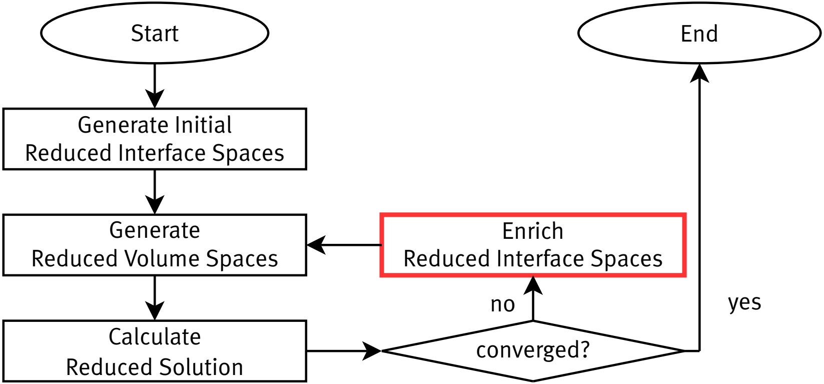
$$||u_\mu - \widetilde{u}_\mu||_V \leq \frac{1}{\alpha_{LB}} \sum c_i ||R_\mu(\widetilde{u}_\mu)||_{O_i'}$$

- It is rigorous and efficient.
- It is online-offline decomposable.
- It is parallelizeable with little amount of communication.
- After a localized geometry change, the offline computed data in unaffected regions can be reused.

Spaces ...

- ... are spanned by FE ansatz functions (for fast evaluation).
- ... have sparse inner product matrix (for fast dual norm calculation).

ArbiLoMod Overview



5. Enrich Interface Spaces

Two main questions:

- Which space to enrich?

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Two main questions:

- Which space to enrich?
- How to enrich?

Which Space to Enrich?

Which Space to Enrich?

Space decomposition into local subspaces

$$V = \bigoplus_i V_i \qquad V_i \subset V$$

Find reduced local spaces

$$\widetilde{V}_i \subset V_i$$

Construct global reduced space

$$\widetilde{V}_{\text{LRB}} := \bigoplus_i \widetilde{V}_i$$

Enrichment

1. Find largest residual:

$$\hat{k}, \hat{\mu} \leftarrow \underset{\mu,k}{\arg\max} \, ||R_\mu(\widetilde{u}_\mu)||_{O'_k}$$

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2. Solve local problem:
Find $u_l \in O_{\hat{k}}$ such that:

$$a(u_l, \varphi; \hat{\mu}) = R_{\hat{\mu}}(\widetilde{u}_{\hat{\mu}})(\varphi) \qquad \forall \varphi \in O_{\hat{k}}$$

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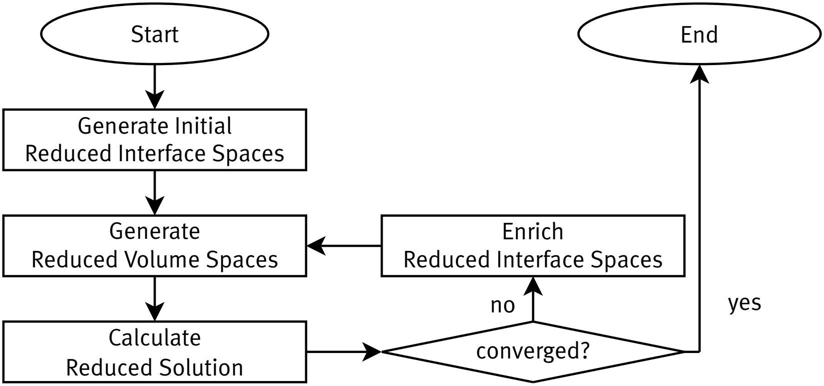
3. Apply space decomposition:

$$\hat{i} \leftarrow \underset{i}{\arg\max} \, || \left(P_{V_i}(u_l)\right)^\perp ||_{V_i}$$

$$\widetilde{V}_{\hat{i}} \leftarrow \widetilde{V}_{\hat{i}} \oplus \operatorname{span}\left(\left(P_{V_i}(u_l)\right)^\perp\right)$$

With $(P_{V_i}(u_l))^\perp$: part of $P_{V_i}(u_l)$ orthogonal to $\widetilde{V}_i$.
Similar to online adaptivity in GMsFEM [Chung et al., 2015].

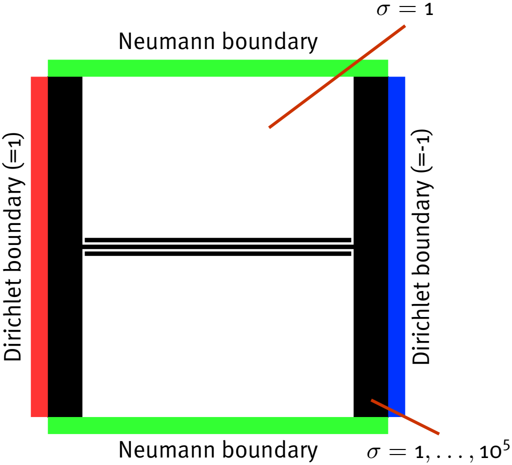
ArbiLoMod Overview



Numerical Example

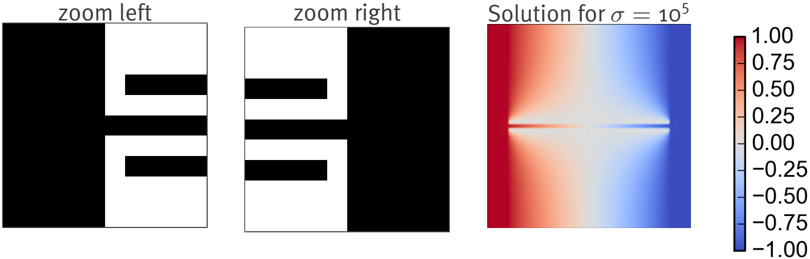
Heat conduction

$-\nabla \cdot (\sigma \nabla u) = 0$



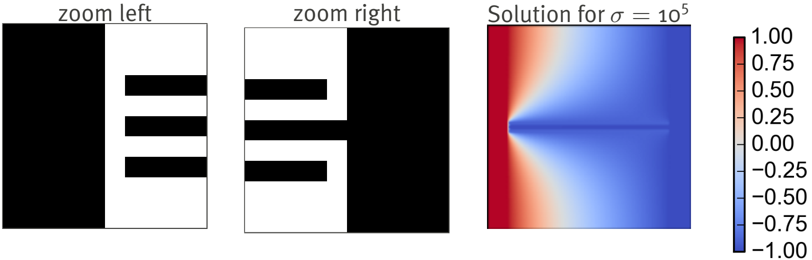
Sequence of Geometries

Geometry 1:



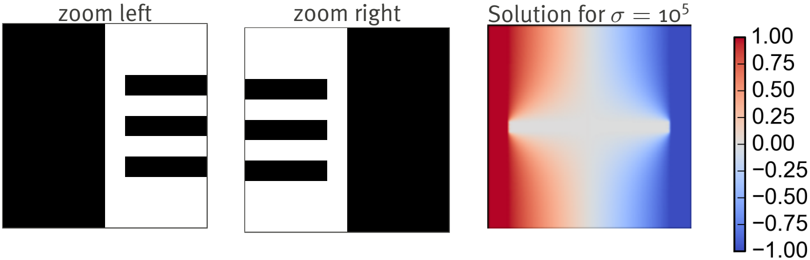
Sequence of Geometries

Geometry 2:



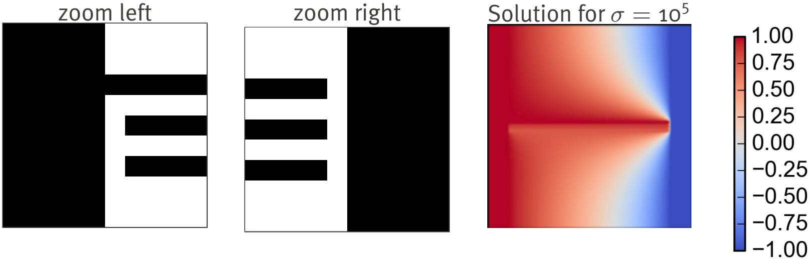
Sequence of Geometries

Geometry 3:



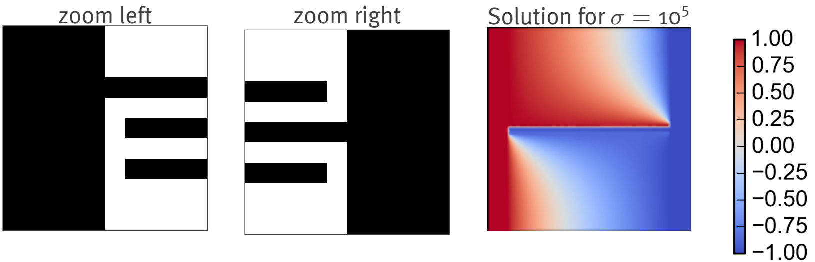
Sequence of Geometries

Geometry 4:



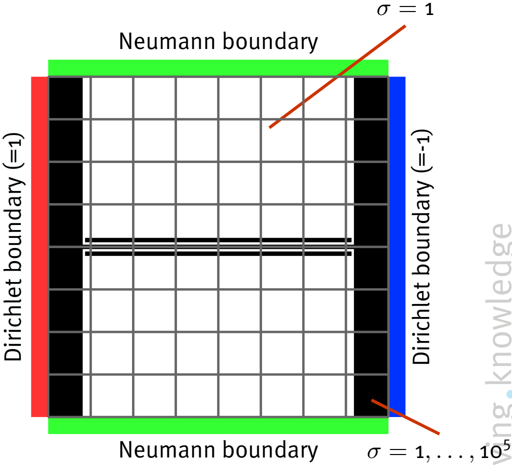
Sequence of Geometries

Geometry 5:

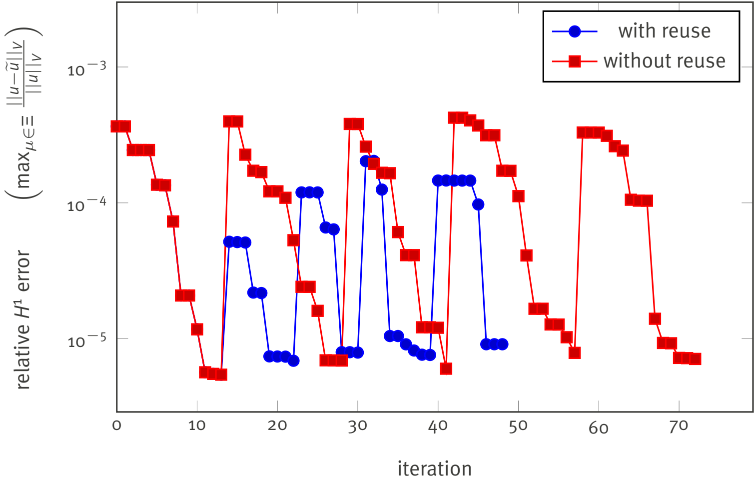


Numerical Example

- Heat conduction
  $-\nabla \cdot (\sigma \nabla u) = 0$
- 8x8 domain decomposition
- 80.401 dofs in full model



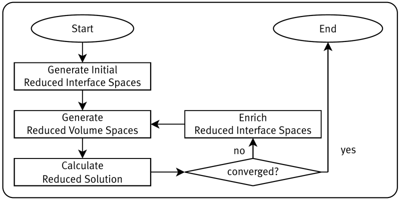
Convergence



Savings

geometry	trainings		greedys		iterations	
	reuse:		reuse:		reuse:	
	no	yes	no	yes	no	yes
1	112	112	64	64	14	14
2	112	5 (-96 %)	64	8 (-88 %)	15	9 (-40 %)
3	112	5 (-96 %)	64	8 (-88 %)	13	8 (-38 %)
4	112	3 (-97 %)	64	6 (-91 %)	16	9 (-44 %)
5	112	5 (-96 %)	64	8 (-88 %)	15	9 (-40 %)

Summary



ArbiLoMod

- ... handles arbitrary local modifications.
- ... intended for interactive use.
- ... is based on the Reduced Basis Method.
- ... has localized and adaptive generation of basis vectors.
- ... is communication avoiding.

Acknowledgements

Many thanks to...

CST - Computer Simulation Technology AG[1]

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[1] www.cst.com

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