Westfälische Wilhelms-Universität Münster

Applied Mathematics Münster

Simulation of Electromagnetic Fields in Highly Complex Printed Circuit Boards using Localized Model Order Reduction

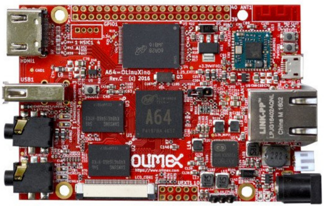
4th Applied Mathematics Symposium Münster

living.knowledge
WWU Münster

Andreas Buhr

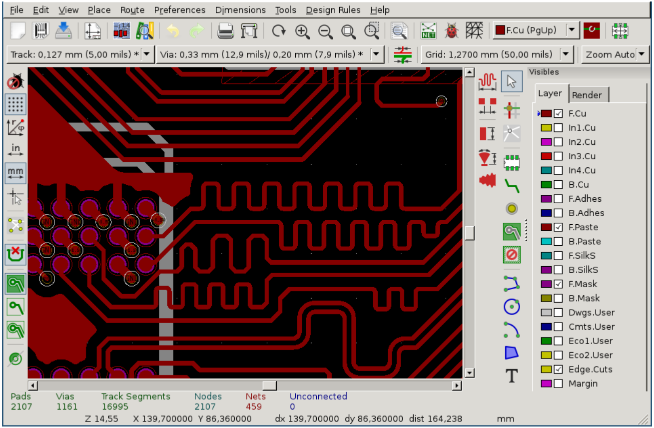
March 16, 2018

Olimex A64



- 1.2 GHz quad-core ARM CPU
- 1 GB of RAM
- open hardware
- designed with KiCAD

KiCAD


File Edit View Place Route Preferences Dimensions Tools Design Rules Help
F.Cu (PgUp)
Track: 0,127 mm (5,00 mils) *
Via: 0,33 mm (12,9 mils)/ 0,20 mm (7,9 mils) *
Grid: 1,2700 mm (50,00 mils)
Zoom Auto
Visibles
Layer
Render
F.Cu
In1.Cu
In2.Cu
In3.Cu
In4.Cu
B.Cu
F.Adhes
B.Adhes
F.Paste
B.Paste
F.SilkS
B.SilkS
F.Mask
B.Mask
Dwgs.User
Cmts.User
Eco1.User
Eco2.User
Edge.Cuts
Margin
Pads 2107
Vias 1161
Track Segments 16995
Nodes 2107
Nets 459
Unconnected 0
Z 14,55 X 139,700000 Y 86,360000 dx 139,700000 dy 86,360000 dist 164,238 mm

Signal Integrity Analysis

Main question:
Do signals arrive at an acceptable quality?

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Bit sequence: 0100110100111110110010

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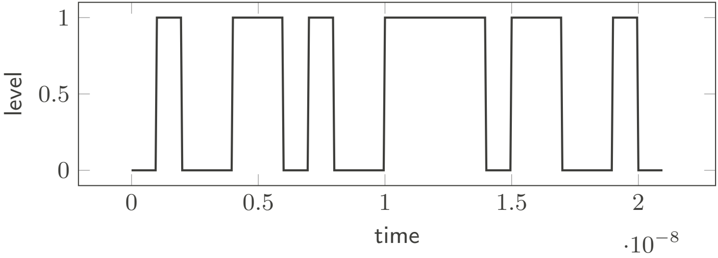
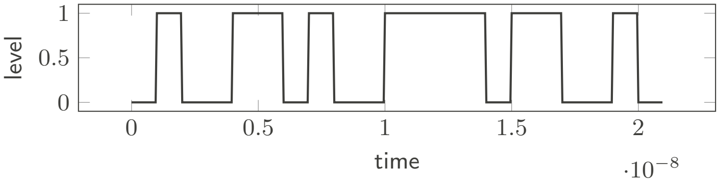
Signal at 1Gbps:

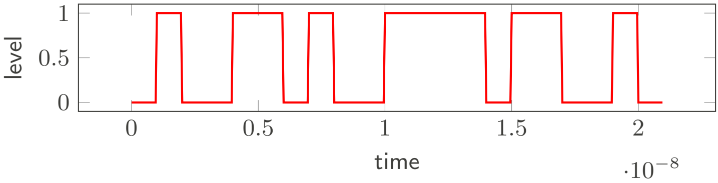


Signal Integrity Analysis

Bit sequence: 01001101001110110010 Signal:

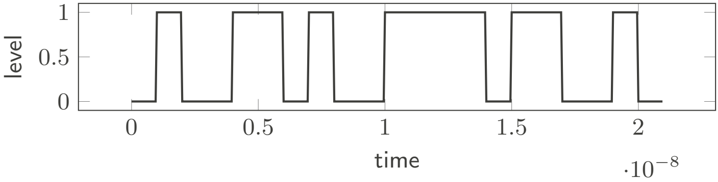


Signal with frequency content < 10 GHz:

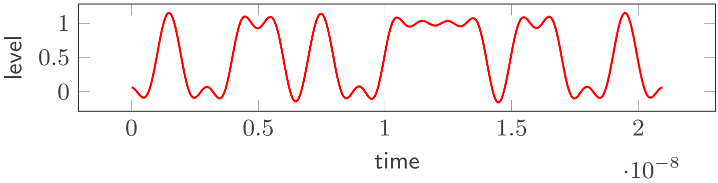


Signal Integrity Analysis

Bit sequence: 01001101001110110010 Signal:

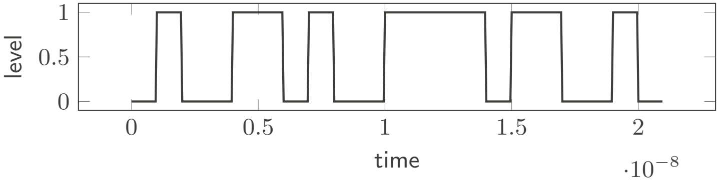


Signal with frequency content < 1 GHz:

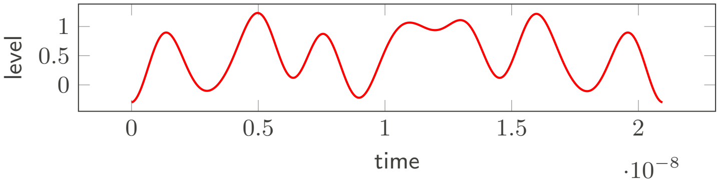


Signal Integrity Analysis

Bit sequence: 010011010011110110010 Signal:

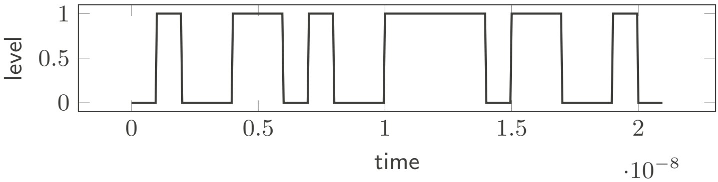


Signal with frequency content < 0.5 GHz:

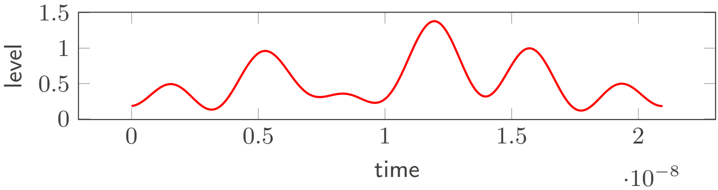


Signal Integrity Analysis

Bit sequence: 010011010011110110010 Signal:



Signal with frequency content < 0.3 GHz:



Time Harmonic Maxwell's Equations

$$\nabla \times \frac{1}{\mu} \nabla \times E - \omega^2 \epsilon E = -i\omega j \qquad \text{in} \quad \Omega$$

- Simulation in a frequency range, e.g.

$$\omega \in [0, 10^{10}]$$

- Dirichlet boundary:

$$E \times n = g \quad \text{on } \partial\Omega \quad (= 0 \text{ on most of } \partial\Omega)$$

- Resulting bilinear and linear form:

$$a(\varphi_1, \varphi_2) = \int_\Omega \frac{1}{\mu} (\nabla \times \varphi_1) \cdot (\nabla \times \varphi_2) - \omega^2 \varepsilon \int_\Omega \varphi_1 \cdot \varphi_2$$

$$f(\varphi) = -i\omega \int_\Omega j \cdot \varphi$$

"Localized Model Order Reduction"

- Setting
  - Domain Ω
  - Function space V, e.g. FE approximation of $H(\mathrm{curl}, \Omega)$ + B.C.
  - (P)PDE in weak formulation

$$a(u, \varphi) = f(\varphi) \qquad \forall \varphi \in V$$

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- Reduction
  - Reduced spaces $\widetilde{V} \subset V$ and $\widetilde{W} \subset V$

$$\widetilde{V} := \mathrm{span}\{\psi_1^a, \dots, \psi_n^a\}$$

$$\widetilde{W} := \mathrm{span}\{\psi_1^t, \dots, \psi_n^t\}$$

  - Reduced solution $\widetilde{u} \in \widetilde{V}$, s.t.

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  - Reduced solution $\widetilde{u} \in \widetilde{V}$, s.t.

$$a(\widetilde{u}, \widetilde{\varphi}) = f(\widetilde{\varphi}) \qquad \forall \widetilde{\varphi} \in \widetilde{W}$$

- Localized
  - supp(ψ_i) much smaller than Ω
  - supp(ψ_i) much larger than meshsize h

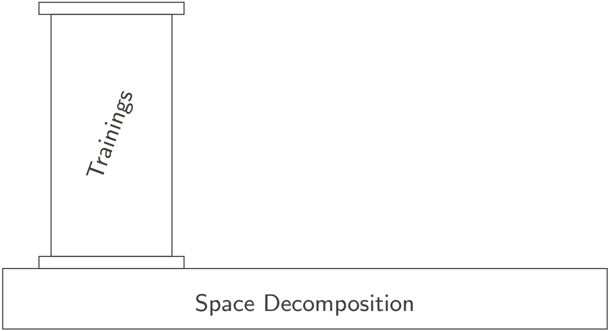
Related Fields

- Reduced Basis Methods
- Multiscale Methods
- Domain Decomposition Methods

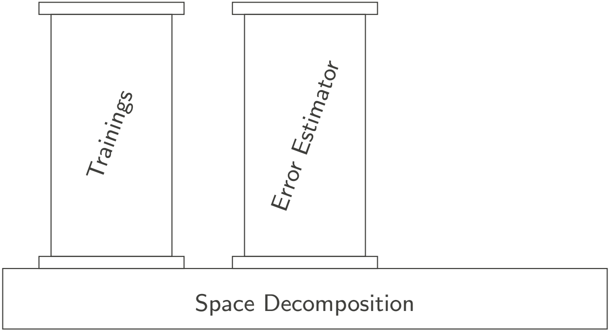
Localized Model Order Reduction

Space Decomposition

Localized Model Order Reduction



Localized Model Order Reduction



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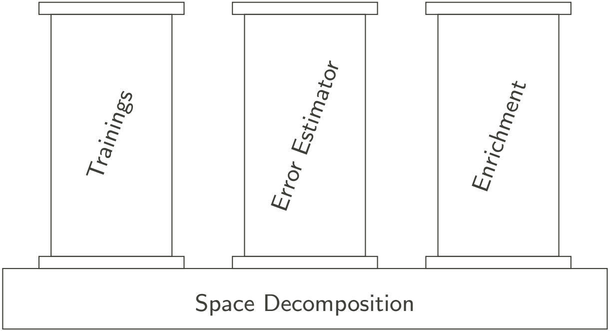
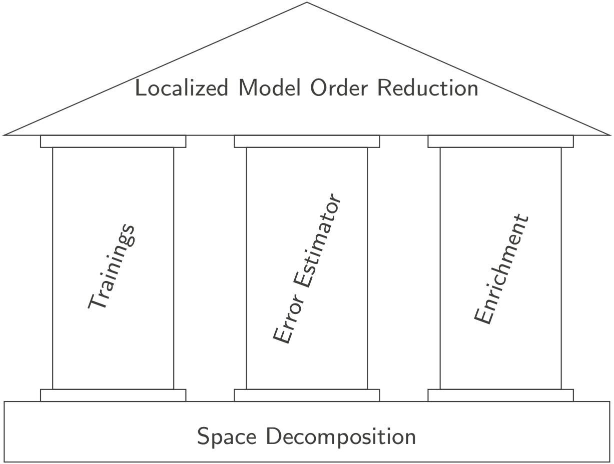
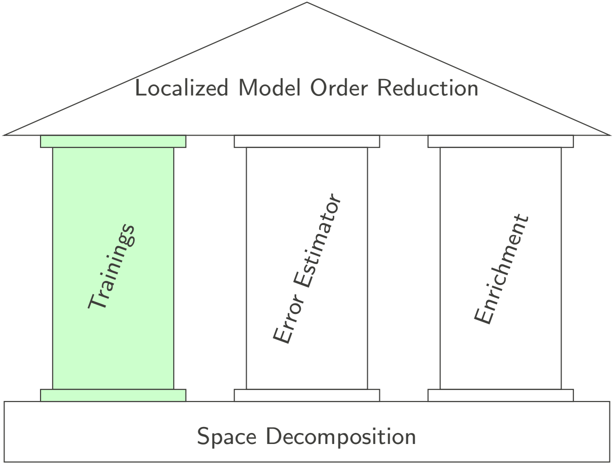


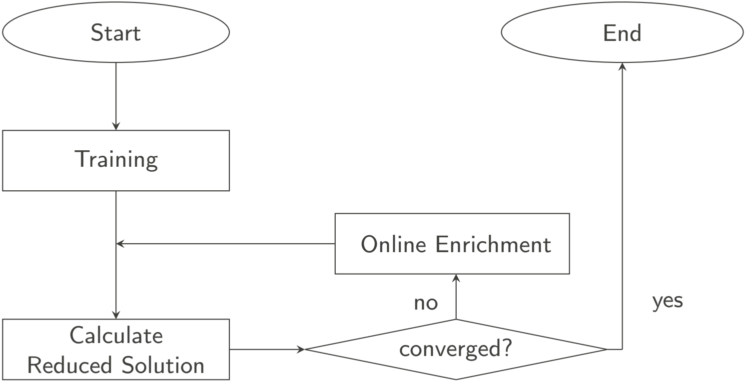
Localized Model Order Reduction



Localized Model Order Reduction



Localized Model Order Reduction



Localized Reduced Basis

Space decomposition into local subspaces

$$V = \sum_i V_i \qquad V_i \text{ subspaces of } V$$

Mappings to local spaces

$$P_{V_i} : V \to V_i$$

$$\sum_i P_{V_i}(\varphi) = \varphi \qquad \forall \varphi \in V$$

$$P_{V_i} \text{ linear}$$

Examples

1. in GFEM [Babuška and Melenk 1997], you could have
   - $V = H_0^1(\Omega)$
   - $V_i = H_0^1(\omega_i)$ on overlapping DD ω_i
   - P_{V_i} is multiplication with suited partition of unity
2. in LRBMS [Ohlberger and Schindler 2015], you could have
   - V is DG-space
   - $V_i = V|_{\omega_i}$ with non-overlapping DD ω_i
   - P_{V_i} is restriction on ω_i
3. in ArbiLoMod [Buhr, Engwer, et al. 2017], you could have
   - $V = H_0^1(\Omega)$
   - $V_i = H_0^1(\omega_i)$ on non-overlapping DD ω_i
     $+$ space for domain-interfaces

Localized Reduced Basis

Space decomposition into local subspaces

$$V = \sum_i V_i \qquad V_i \text{ subspaces of } V$$

Find reduced local spaces

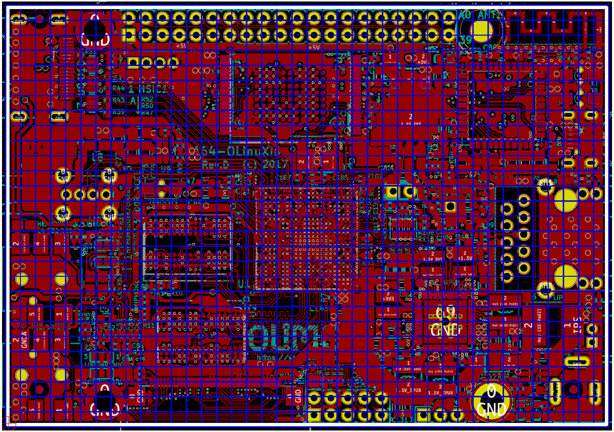
$$\widetilde{V}_i \subset V_i$$

Construct global reduced space

$$\widetilde{V} := \sum_i \widetilde{V}_i$$

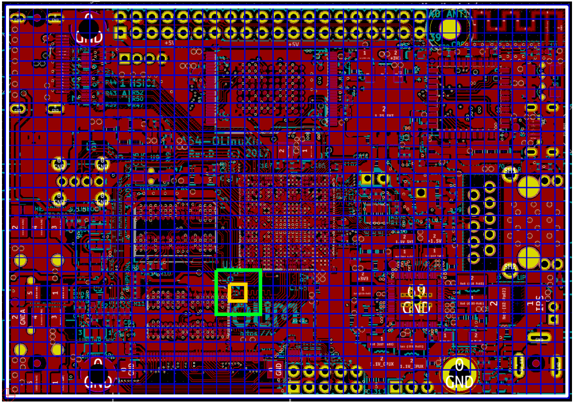
Domain Decomposition

Domain Decomposition with $40 \times 28 = 1120$ domains:



Domain Decomposition

For each domain ω_i (e.g. ω_{816}, yellow)
we introduce an oversampling domain ω_i^* (e.g. ω_{816}^*, green)



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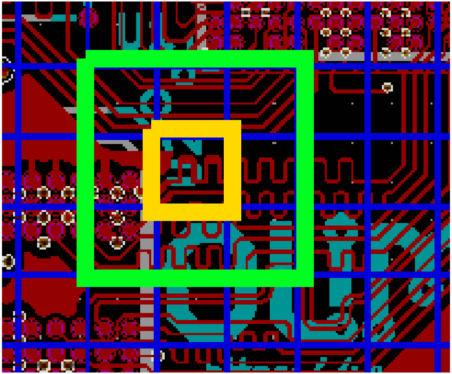

Transfer Operators

Following [Smetana and Patera 2016] we introduce transfer operators mapping from $\partial\omega_i^*$ to ω_i.

- Spaces of a-harmonic functions:

$$\mathcal{H}_i := \{w \in H(\mathrm{curl};\omega_i^*) \;:\; a(w,\varphi) = 0 \quad \forall \varphi \in H_0(\mathrm{curl};\omega_i^*)\}$$

- Source spaces:

$$S_i := \{w|_{\partial\omega_i^*} \;:\; w \in \mathcal{H}_i\}$$

- Range spaces:

$$R_i := \{w|_{\omega_i} \;:\; w \in \mathcal{H}_i\}$$

- Transfer operators $T_i : S_i \to R_i$ defined by:

$$T_i(w|_{\partial\omega_i^*}) = w|_{\omega_i} \qquad \forall w \in \mathcal{H}_i$$

Why are transfer operators interesting?

The reason is

$$u|_{\omega_i} = T_i(u|_{\partial\omega_i^*}) + u_{f,i}$$

where $u_{f,i}$ is the part induced by the right hand side;
$u_{f,i} \in H_0(\mathrm{curl};\omega_i^*)$ is defined as the solution of

$$a(u_{f,i},\varphi) = f(\varphi) \qquad \forall \varphi \in H_0(\mathrm{curl};\omega_i^*).$$

- $u_{f,i}$ is cheap to compute
- $T_i(u|_{\partial\omega_i^*})$ is in the image of T_i.

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- $u_{f,i}$ is cheap to compute
- $T_i(u|_{\partial\omega_i^*})$ is in the image of T_i.

If we can approximate the image of T_i,
we can approximate the solution on ω_i.

New goal

Generate subspaces $\widetilde{R}_i \subset R_i$ so that

$$\left\| T_i - P_{\widetilde{R}_i} T_i \right\|$$

is small.
$P_{\widetilde{R}_i} : R_i \to \widetilde{R}_i$ is orthogonal projection.

SVD of T_i

drop index i for now...

SVD of T

If T is compact, it has a singular value decomposition

$$T(\zeta) = \sum_j \phi_j \sigma_j (\chi_j, \zeta)_S \qquad \forall \zeta \in S$$
$$(\phi_j, \phi_k)_R = \delta_{j,k}$$
$$(\chi_j, \chi_k)_S = \delta_{j,k}$$

its left singular values form the optimal $\widetilde{R}$.
With $\widetilde{R}_n := \operatorname{span}\{\phi_1, \ldots, \phi_n\}$, it holds

$$\left\| T - P_{\widetilde{R}_n} T \right\| = \sigma_{n+1}$$

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Optimality

optimal to approximate the image of T
not optimal to approximate the solution u

Randomized Range Finder

In [Buhr and Smetana 2017] we applied methods from randomized linear algebra [Halko, Martinsson, and Tropp 2011] to this problem and obtained:
With

$$\widetilde{R}_n := \operatorname{span}\{T D_S^{-1} \underline{r}_1, \ldots, T D_S^{-1} \underline{r}_n\}$$

- $\underline{r}_i$: i.i.d. random vector with normal distributed entries in $\mathbb{R}^{\dim(S)}$.
- D_S^{-1}: ritz isomorphism $\mathbb{R}^{\dim(S)} \to S$.

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it holds

$$\mathbb{E}\|T - P_{R^n}T\| \leq \sqrt{\frac{\lambda_{max}^{M_R}}{\lambda_{min}^{M_R}} \frac{\lambda_{max}^{M_S}}{\lambda_{min}^{M_S}}} \min_{\substack{k+p=n \\ k\geq 2, p\geq 2}} \left[\left(1 + \sqrt{\frac{k}{p-1}}\right) \sigma_{k+1} + \frac{e\sqrt{n}}{p} \left(\sum_{j>k} \sigma_j^2 \right)^{\frac{1}{2}} \right].$$

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$$\approx c \cdot \sqrt{n}\sigma_n$$

Randomized a posteriori error estimator

For any $\widetilde{R}$, it holds

$$\left\|T - P_{\widetilde{R}}T\right\| \leq c_{\text{est}}(n_t, \varepsilon_{\text{testfail}}) \max_{i \in 1,\dots,n_t} \left\|(T - P_{\widetilde{R}}T)\, D_S^{-1}\, \underline{r}_i\right\|_R$$

with a probability greater than $(1 - \varepsilon_{\text{testfail}})$.

Randomized a posteriori error estimator

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$$c_{\mathrm{est}}(n_t, \varepsilon_{\mathrm{testfail}}) := \left[\sqrt{2\lambda_{min}^{M_S}}\ \mathrm{erf}^{-1}\left(\sqrt[n_t]{\varepsilon_{\mathrm{testfail}}}\right)\right]^{-1},$$

$\underline{r}_i$ are random normal vectors, and $\lambda_{min}^{M_S}$ is the smallest eigenvalue of the matrix of the inner product in S.

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Theory Summary

1. Optimal spaces:

$$\sigma_{n+1} \leq \left\| T - P_{\widetilde{R}_n} T \right\| \qquad \text{for all } \widetilde{R}_n \text{ of size } n$$

2. For randomized created spaces:

$$\left\| T - P_{\widetilde{R}_n} T \right\| \text{ behaves like } \sqrt{n}\sigma_n$$

3. Error estimator

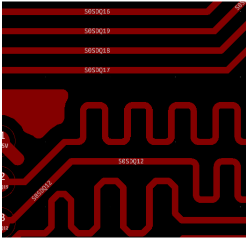
$$\left\| T - P_{\widetilde{R}} T \right\| \leq c_{\mathrm{est}}(n_t, \varepsilon_{\mathrm{testfail}}) \max_{i \in 1,\dots,n_t} \left\| (T - P_{\widetilde{R}} T)\, D_S^{-1}\, \underline{r}_i \right\|_R$$

Olimex A64 Discretization

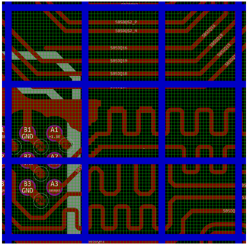
- Polygon description has 273.092 vertices
- PCB dimensions: 90.0mm $\times$ 62.5mm
- meshed with hexahedrons of max. $95\mu\text{m}$ in each dimension
- 960 meshcells in x-direction,
  673 meshcells in y-direction,
  33 meshcells in z-direction.
- $\approx$ 21 million meshcells,
  $\approx$ 65 million DOFs

Staircase Meshing

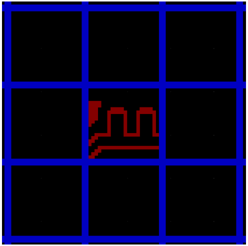
True Geometry



Mesh+DD

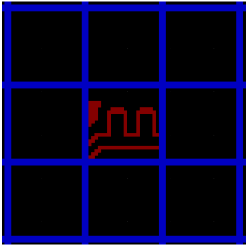


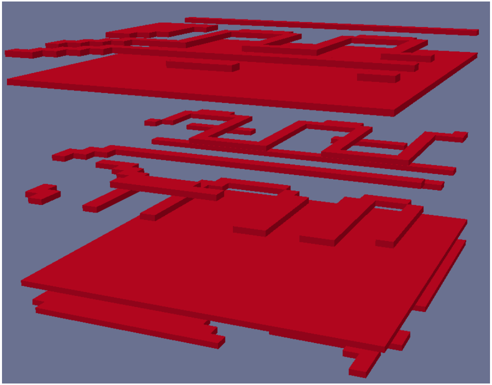
Staircase (one dommain)



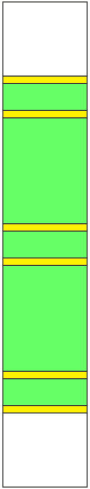
We are only interested in MOR behavior, so we accept this modelling error.

Staircase Meshing



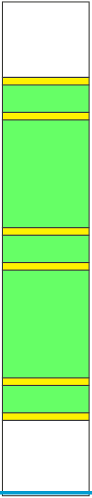


Stackup



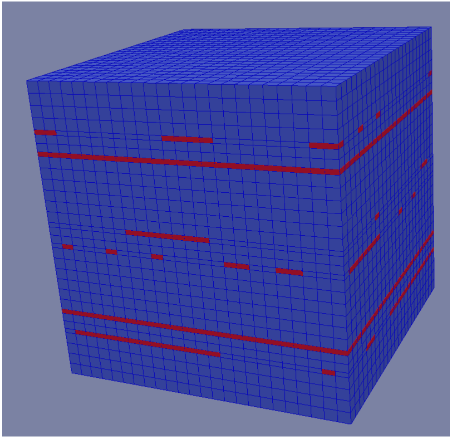
air	350 μm
copper	35 μm
prepreg	127 μm
copper	35 μm
FR4	500 μm
copper	35 μm
prepreg	127 μm
copper	35 μm
FR4	500 μm
copper	35 μm
prepreg	127 μm
copper	35 μm
air	350 μm

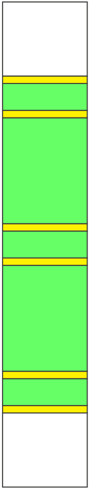
Stackup with z-Mesh



air	350 μm	4 meshcells
copper	35 μm	1 meshcell
prepreg	127 μm	2 meshcells
copper	35 μm	1 meshcell
FR4	500 μm	6 meshcells
copper	35 μm	1 meshcell
prepreg	127 μm	2 meshcells
copper	35 μm	1 meshcell
FR4	500 μm	6 meshcells
copper	35 μm	1 meshcell
prepreg	127 μm	2 meshcells
copper	35 μm	1 meshcell
air	350 μm	4 meshcells

Stackup





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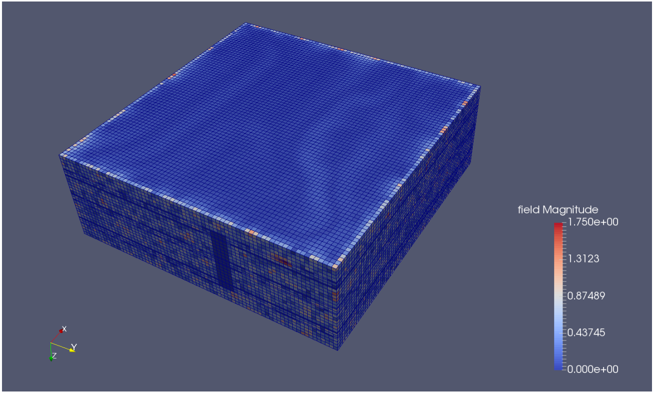
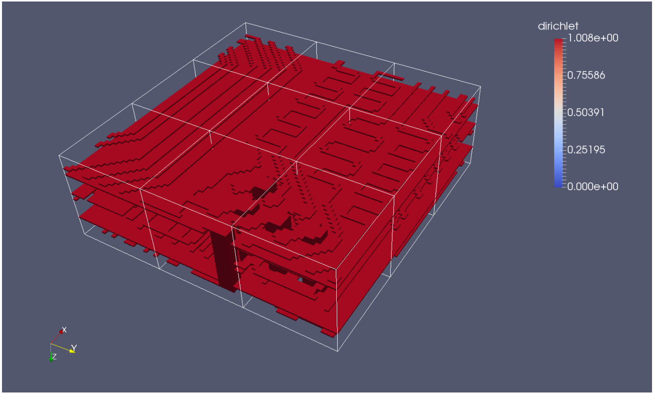
prepreg	127 μm
copper	35 μm
FR4	500 μm
copper	35 μm
prepreg	127 μm
copper	35 μm
FR4	500 μm
copper	35 μm
prepreg	127 μm
copper	35 μm
air	350 μm

Example field for random boundary data

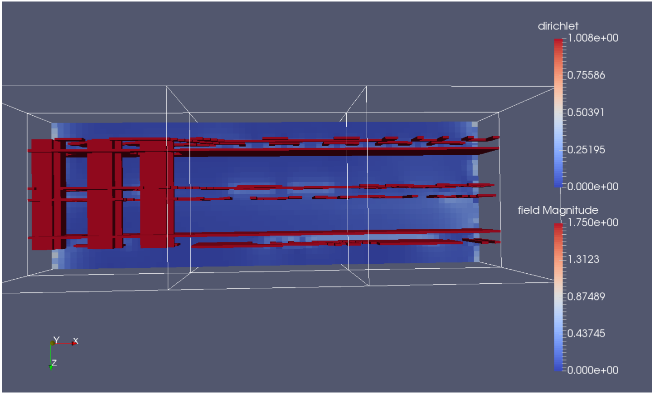
(around 350.000 unknowns)



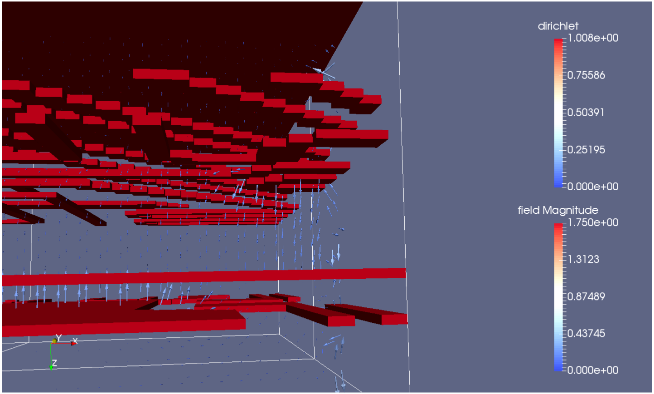
Example field for random boundary data



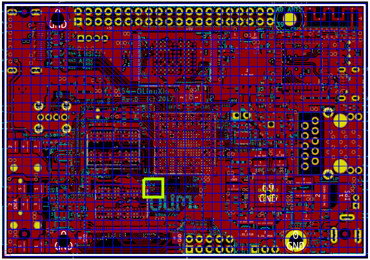
Example field for random boundary data

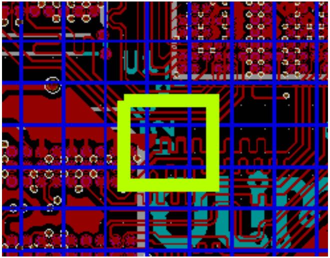


Example field for random boundary data

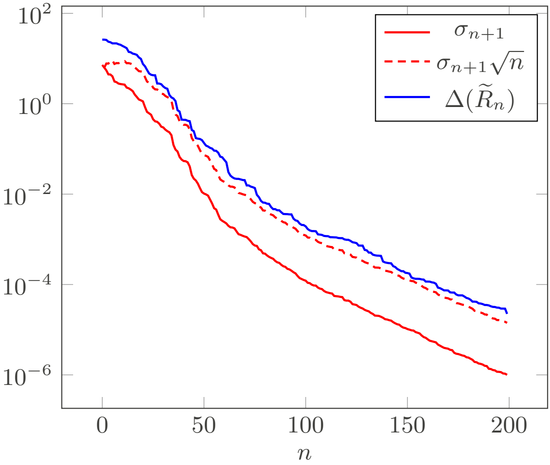


Domain 816

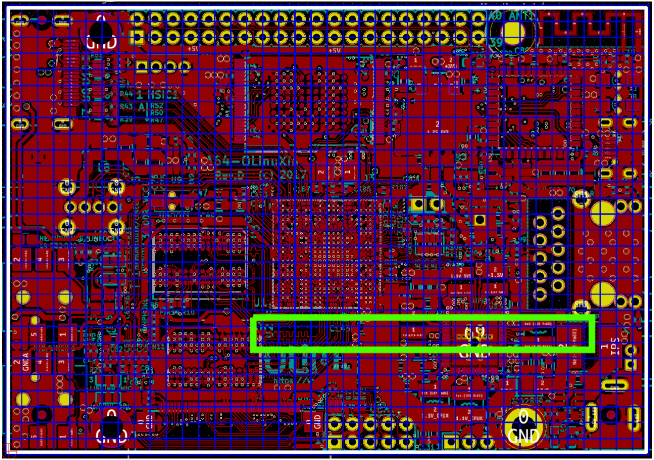




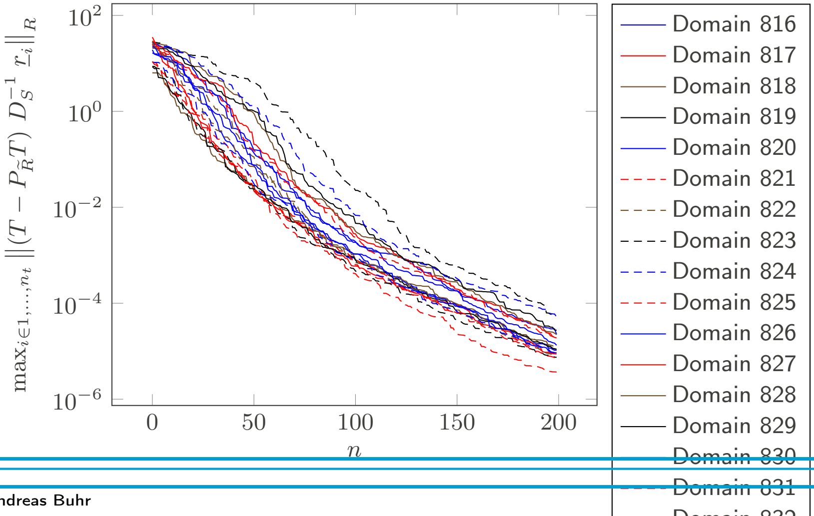
Decay of Singular Values for T_{816}



Error Estimator Decay



Error Estimator Decay



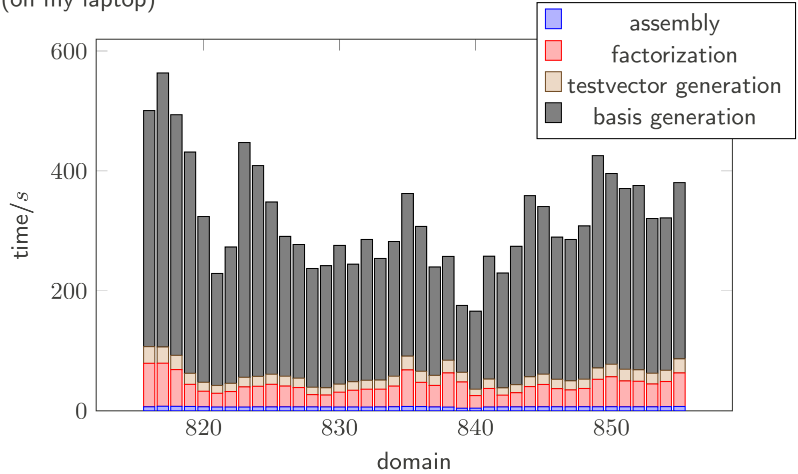
Adaptive Runs

Adaptive run till

$$\max_{i \in 1,\dots,n_t} \left\| (T - P_{\widetilde{R}} T)\; D_S^{-1}\; \underline{r}_i \right\|_R < 10^{-2}$$

CPU timings

(on my laptop)



Reduction

Domain	unreduced DOFs	reduced DOFs	factor
816	394422	106	3721
817	399921	121	3305
818	395745	116	3412
819	385383	120	3212
820	365149	107	3413
821	345178	88	3922
822	343729	95	3618
823	348205	138	2523
824	359842	127	2833
825	353213	107	3301
826	358154	94	3810
827	354729	88	4031
828	358521	91	3940
829	358206	94	3811
830	344408	97	3551
...	...	...	...

Summary

- Randomized range finder was applied to PCB geometry
- Within 5 to 10 minutes per domain, it is possible to create reduced spaces of dimension about 100, which are of acceptable quality.